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Contributions dans l'analyse des modèles vectoriels de séries chronologiques saisonnières et périodiques

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DEDICATION

À la mémoire de mon père Ion Ursu

SOMMAIRE

Les séries chronologiques avec des propriétés saisonnières ou périodiques sont utilisées dans des domaines divers comme la météorologie, l'économie ou l'hydrologie. L'étude des séries chronologiques périodiques a reçu beaucoup d'attention dans les dernières années. La plus grande partie de la littérature existante traite des modèles périodiques univariés. Cependant, les modèles multivariés sont potentiellement plus utiles dans la pratique, puisque la plupart des situations réelles impliquent plusieurs variables et des séries chronologiques multidimensionnelles.

Dans le présent travail, on s'intéresse à la modélisation et l'ajustement des modèles de séries chronologiques présentant une structure autorégressive périodique vectorielle et des séries chronologiques qui offrent une structure vectorielle autorégressive moyenne mobile saisonnière. La thèse est présentée sous la forme de trois articles scientifiques.

Dans le premier article, nous considérons le modèle saisonnier vectoriel autorégressif moyenne mobile, dont l'abréviation est l'acronyme SVARMA, pour modéliser des données saisonnières multidimensionnelles. Nous discutons de l'estimation par moindres carrés des paramètres du modèle, en permettant aux paramètres de satisfaire des contraintes linéaires. La distribution asymptotique des matrices d'autocovariance résiduelles est obtenue. Afin de vérifier l'adéquation du modèle, des statistiques de test portemanteaux sont considérées et les distributions asymptotiques sont étudiées. Nous utilisons ensuite plusieurs modèles saisonniers simulés afin d'illustrer le comportement du test proposé.

Dans le deuxième article, nous trouvons la distribution asymptotique des estimateurs des paramètres dans un modèle vectoriel périodique (PVAR). Nous permettons aux paramètres dans une saison donnée de satisfaire des contraintes

linéaires. Afin de vérifier l'adéquation du modèle, nous utilisons des tests basés sur les autocovariances ou les autocorrélations résiduelles. La distribution asymptotique des matrices d'autocovariances et autocorrélations résiduelles est établie. Des tests de type portemanteau sont introduits et leur distribution asymptotique est étudiée.

Dans le troisième article, nous introduisons une famille de modèles de séries chronologiques saisonnières avec des paramètres qui varient avec la saison (SPVAR). Ce modèle autorégressif combine les séries chronologiques périodiques (PVAR) et les séries chronologiques saisonnières multiplicatives (SVAR). Une vue d'ensemble sur la construction du modèle SPVAR est donnée en soulignant les trois étapes du développement du modèle : conditions de stationnarité et calcul des autocovariances, estimation par moindres carrés et propriétés asymptotiques des estimateurs, vérification diagnostique en exploitant la distribution asymptotique des matrices d'autocovariances résiduelles. On étudie la distribution asymptotique du test diagnostique sous l'hypothèse nulle. Des résultats de simulations sont ensuite présentés afin d'illustrer le comportement du test proposé.

MOTS CLÉS :

Test diagnostique ; série chronologique périodique ; statistiques portemanteaux ; matrices d'autocorrélations et d'autocovariances résiduelles ; série chronologique saisonnière ; série chronologique multivariée.

SUMMARY

Time series with seasonal or periodic properties naturally arise in many fields, such as climatology, economics or hydrology. The study of periodic time series has received much attention in recent years. Most of the existing literature deals with periodic univariate models. However, multivariate models are expected to be more useful in practice, since most real situations involve several variables and vector time series.

In this work, we are interested in modeling and adjustment of vector periodic autoregressive time series models and seasonal vector autoregressive moving average models. The thesis is presented in the form of three scientific articles.

In the first article, we consider seasonal vectorial autoregressive moving average models, abbreviated by the acronym SVARMA, to describe seasonal data. We discuss least squares estimation of the model parameters, allowing to satisfy linear constraints. The asymptotic distributions of the residual autocovariance matrices in the class of SVARMA time series models are obtained. In order to check model adequacy, portmanteau test statistics are considered and their asymptotic distributions are studied. We simulate several seasonal models to illustrate the proposed test statistic.

In the second article we derive the asymptotic distributions of the estimators of the model parameters in PVAR model, allowing the parameters in a given season to satisfy linear constraints. To verify the adequacy of the model, we use test statistics based on residual autocovariances or residual autocorrelation. We find the asymptotic distribution of the residual autocovariances matrices for PVAR models. Portmanteau tests statistics are introduced and we study their asymptotic distributions.

In the third section, we introduce a class of multivariate seasonal time series with periodically varying parameters (SPVAR). This model combines the vector periodic autoregressive time series models (PVAR) and the multiplicative seasonal time series (SVAR). An overview of the construction of the SPVAR model is given, emphasizing three stages of development : stationary conditions and autocovariance matrices, least squares estimation and asymptotic distributions of this estimators, diagnostic checking based on asymptotic distributions of the residual autocovariance matrices. In order to check model adequacy, portmanteau test statistics are considered and their asymptotic distributions are studied. A small simulation study is conducted to investigate the finite-sample properties of the proposed test statistics.

KEY WORDS :

Diagnostic checking; periodic time series; portmanteau test statistics; residual autocorrelation and autocovariance matrices; seasonal time series; vector time series.

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Chapitre 1

INTRODUCTION

Les séries chronologiques avec des propriétés saisonnières ou périodiques sont utilisées dans de nombreux domaines, tels que la climatologie, l'hydrologie et l'économie, entre autres. Dans cette thèse on s'intéresse principalement aux processus vectoriels saisonniers multiplicatifs autorégressifs moyennes mobiles (SVARMA) et aux processus vectoriels autorégressifs périodiques (PVAR). Un nouveau modèle est également proposé, combinant les modèles SVAR et PVAR, que nous notons SPVAR.

Les modèles de séries chronologiques univariés saisonnières, tels que les modèles présentant une structure saisonnière autorégressive moyenne mobile intégrée (SARIMA) développés à l'origine par Box et Jenkins (1970), ont été largement étudiés dans la littérature. Ce sont des modèles non stationnaires qui peuvent être transformés en processus ARMA stationnaires après l'application de certains filtres. La caractéristique commune des séries saisonnières est une période connue s (par exemple $s = 4$ correspond à des données trimestrielles, et $s = 12$ à des données mensuelles). Contrairement au cas univarié, les modèles vectoriels saisonniers multiplicatifs ont été moins étudiés, ce qui n'est guère surprenant compte tenu de la plus grande complexité de ces modèles. Reinsel (1997, p. 219) présente le modèle vectoriel saisonnier multiplicatif autorégressif moyenne mobile (SVARMA).

Soit $\mathbf{Y} = \{\mathbf{Y}_t, t \in \mathbb{Z}\}$, un processus stochastique, où $\mathbf{Y}_t = (Y_t(1), \ldots, Y_t(d))^\top$ est un vecteur de dimension d. Soit B et B^s l'opérateur retard usuel et l'opérateur retard saisonnier, respectivement, où s représente la période. Le processus

stochastique $\mathbf{Y}$ est un processus multiplicatif saisonnier autorégressif moyenne mobile, noté par SVARMA$(p,q) \times (P,Q)_s$, s'il satisfait :

$$\mathbf{\Lambda}(B^s)\mathbf{\Phi}(B)\mathbf{Y}_t = \mathbf{\Xi}(B^s)\mathbf{\Theta}(B)\boldsymbol{\epsilon}_t,\ t \in \mathbb{Z}. \tag{1.1}$$

Les polynômes non saisonniers AR et MA sont définis par $\mathbf{\Phi}(B) = \mathbf{I}_d - \mathbf{\Phi}_1 B - \ldots - \mathbf{\Phi}_p B^p$ et $\mathbf{\Theta}(B) = \mathbf{I}_d - \mathbf{\Theta}_1 B - \ldots - \mathbf{\Theta}_q B^q$, respectivement, tandis que les polynômes saisonniers AR et MA sont donnés par $\mathbf{\Lambda}(B^s) = \mathbf{I}_d - \mathbf{\Lambda}_1 B^s - \ldots - \mathbf{\Lambda}_P B^{sP}$ et $\mathbf{\Xi}(B^s) = \mathbf{I}_d - \mathbf{\Xi}_1 B^s - \ldots - \mathbf{\Xi}_Q B^{sQ}$, respectivement. La matrice $\mathbf{I}_d$, de dimension $d \times d$, représente la matrice identité de dimension d. Le processus d'erreur $\boldsymbol{\epsilon} = \{\boldsymbol{\epsilon}_t, t \in \mathbb{Z}\}$, $\boldsymbol{\epsilon}_t = (\epsilon_t(1), \ldots, \epsilon_t(d))^\top$, correspond à un bruit blanc de moyenne zéro, c'est-à-dire $\boldsymbol{\epsilon}$ est composé des vecteurs aléatoires indépendants tel que $E(\boldsymbol{\epsilon}_t) = \mathbf{0}$ et $E(\boldsymbol{\epsilon}_t \boldsymbol{\epsilon}_t^\top) = \mathbf{\Sigma}_{\boldsymbol{\epsilon}}$, où $\mathbf{\Sigma}_{\boldsymbol{\epsilon}}$ est non singulière. Les coefficients des polynômes saisonniers et non saisonniers sont des matrices $d \times d$, qui, généralement, ne commutent pas. On suppose que le processus SVARMA est causal et inversible. Les conditions de causalité sont $\det\{\mathbf{\Phi}(z)\} \neq 0$ et $\det\{\mathbf{\Lambda}(z^s)\} \neq 0$ pour tout z avec $|z| \leq 1$ (où $\det\{\mathbf{A}\}$ représente le déterminant de la matrice $\mathbf{A}$). De même, les conditions d'inversibilité sont $\det\{\mathbf{\Theta}(z)\} \neq 0$ et $\det\{\mathbf{\Xi}(z^s)\} \neq 0$ pour $|z| \leq 1$.

Comme dans le cas d'un modèle saisonnier multiplicatif univarié (voir Brockwell et Davis (2002, p. 203)), le modèle (1.1) peut être vu tout simplement comme un modèle VARMA$(p + Ps, q + Qs)$ dans lequel les coefficients satisfont des contraintes multiplicatives. Dans le cas univarié, les méthodes pour déterminer les ordres des polynômes saisonniers et non saisonniers sont basées sur le calcul des autocorrélations (ACF) et autocorrélations partielles (PACF) du processus. Dans le cas multivarié, Wei (2006, p. 401) suggère que, généralement, l'identification du modèle est similaire au cas univarié et en conséquence basée sur l'utilisation des matrices d'autocovariances et autocovariances partielles. La sélection finale d'un modèle VARMA peut être basée sur le critère d'information d'Akaike (AIC) (voir Reinsel (1997, p. 160)). Toutefois, comme indiqué par Reinsel (1997, p. 224), il semble nécessaire d'obtenir des contributions supplémentaires dans l'étude des modèles de séries chronologiques vectorielles multiplicatives saisonnières.

Le processus défini en (1.1) représente un modèle VARMA d'ordre $(p+Ps, q+Qs)$ dans lequel certains coefficients sont nuls et les autres sont fonctions du vecteur des paramètres des polynômes saisonniers et non saisonniers. Pour un modèle SARMA, Brockwell et Davis (2002, p. 206) suggèrent l'utilisation du modèle ARMA$(p+Ps, q+Qs)$ sans contraintes multiplicatives dans une première étape. Ensuite, des contraintes sur les coefficients non significatifs sont imposées. Le modèle SARMA est souvent plus utile et possède moins de paramètres $(p+q+P+Q)$ que le modèle ARMA sans contrainte $(p+qs+P+Qs)$. Dans le cas vectoriel, cette approche peut s'avérer problématique étant donné le grand nombre de paramètres à estimer dans une première étape. Les paramètres des polynômes $\mathbf{\Phi}(B)$, $\mathbf{\Theta}(B)$, $\mathbf{\Lambda}(B)$, $\mathbf{\Xi}(B)$ peuvent être estimés en utilisant le maximum de vraisemblance exacte. Dans la littérature on retrouve souvent deux méthodes pour calculer la vraisemblance exacte d'un modèle VARMA : la décomposition de Choleski et le filtre de Kalman. Par exemple, la décomposition de Choleski est utilisée dans Jonasson (2008) et Jonasson et Ferrando (2008). Des modèles plus complexes peuvent être estimés en utilisant le filtre de Kalman comme dans Casals, Sotoca et Jerez (1999) et Terceiro (1990).

De façon générale, l'analyse des modèles saisonniers vectoriels utilise les mêmes étapes décrites dans Box et Jenkins (1970, Chapitre 9). Il est à préciser que dans l'analyse des séries saisonnières un filtre double est souvent préconisé : un filtre saisonnier $(\mathbf{I}_d - B^s)$ (où B est l'opérateur retard) et le filtre de premier ordre $(\mathbf{I}_d - B)$. Ce type de filtre double appliqué aux données originales est utilisé par Box et Jenkins (1976) comme première étape dans l'identification des modèles SARIMA. L'introduction des tests sur les racines unitaires permet de déduire que de manière générale le filtre saisonnier est le seul vraiment nécessaire. Voir les résultats obtenus par Osborn (1990), Beaulieu et Miron (1993) et Franses (1996). Beaucoup de séries chronologiques d'intérêt pratique affichent un comportement saisonnier et il semble important d'avoir des tests diagnostiques permettant de vérifier l'adéquation du modèle proposé. Les tests diagnostiques pour l'adéquation du modèle peuvent être étudiés en utilisant les matrices d'autocovariances résiduelles, entre autres. Les tests portemanteaux pour les modèles VARMA ont

été étudiés par beaucoup d'auteurs, comme Hosking (1980), Li et McLeod (1981) et Poskitt et Tremayne (1982). Ces tests ont été moins étudiés dans le contexte des séries chronologiques saisonnières. McLeod (1978) a obtenu la distribution des autocorrélations résiduelles dans les modèles SARMA.

Les modèles saisonniers ARIMA décrivent une forme simplifiée de saisonnalité, plus précisément on est assuré d'obtenir la stationnarité de la série saisonnière après l'application d'un certain filtre (dans cette thèse, la stationnarité signifie stationnarité au sens large). Toutefois, de nombreuses séries chronologiques saisonnières ne peuvent pas être filtrées afin d'atteindre la stationnarité, la raison étant que la structure des corrélations de ces séries chronologiques dépend de la saison (Tiao et Grupe (1980)). En effet, les méthodes d'ajustement saisonnier traitent les observations dans tous les saisons de la même façon. En conséquence, on peut encore trouver des traces de périodicité dans les données ajustées (voir la section 3.5 de Franses et Paap (2004) pour plus de détails sur cette question). Par exemple, dans les données hydrologiques concernant le débit des fleuves, il est attendu qu'un fort débit au printemps sera observé et un plus faible débit l'été. Ainsi, la corrélation entre les mois correspondants au printemps peut être fort différente de la corrélation entre les mois d'été (voir Vecchia (1985a), Vecchia (1985b) ou McLeod (1993)).

Une famille de modèles qui permet de décrire ce geure de séries chronologiques saisonnières est la famille des modèles périodiques. Pour les séries chronologiques périodiques univariées, les premiers travaux dans la littérature statistique remontent à Jones et Brelsford (1967), Pagano (1978), Vecchia (1985a, 1985b), Vecchia et Ballerini (1991), McLeod (1993) et McLeod (1994), entre autres. Plus récemment, Lund et Basawa (2000) ont exploré les techniques d'évaluation par vraisemblance pour les modèles périodiques autorégressifs moyennes mobiles (PARMA) et Basawa et Lund (2001) ont étudié les propriétés asymptotiques des estimateurs des moindres carrés pondérés de paramètres du modèle. L'introduction des modèles périodiques dans la littérature économique remonte à Parzen et Pagano (1979) et Osborn (1988), entre autres. Beaucoup de séries chronologiques macro-économiques affichent des tendances ; les modèles périodiques

pour des données avec des tendances et la méthodologie de test ont été mis au point par Boswijk et Franses (1996) et Paap et Franses (1999). Voir aussi la monographie de Franses et Paap (2004).

Un modèle autorégressif périodique multivarié (PVAR) généralise le modèle autorégressif (VAR) classique en permettant aux paramètres autorégressifs de varier avec la saison. En autres mots, le modèle PVAR suppose que les observations dans une certaine saison peuvent être décrites par des modèles différents. Le processus stochastique $\mathbf{Y}$ est un processus stochastique autorégressif périodique multivarié (PVAR) si :

$$\mathbf{Y}_{ns+\nu} = \sum_{k=1}^{p(\nu)} \mathbf{\Phi}_k(\nu)\mathbf{Y}_{ns+\nu-k} + \boldsymbol{\epsilon}_{ns+\nu}, \tag{1.2}$$

où pour ν fixé et une valeur s donnée, le vecteur aléatoire $\mathbf{Y}_{ns+\nu}$ représente la réalisation dans la saison ν, avec $\nu \in \{1, \ldots, s\}$, de l'année $n+1$, $n \in \mathbb{Z}$. L'écriture de l'indice t sous la forme $ns+\nu$ nous permet de faire ressortir la saison ν et est aisé à manipuler. Par exemple, dans le cas de données mensuelles où $s = 12$ il est commode d'associer l'observation 32 à la 8-ème saison de l'année 3. L'ordre du modèle autorégressif à la saison ν est donné par $p(\nu)$, et $\mathbf{\Phi}_k(\nu) = (\Phi_{k,ij}(\nu))_{i,j=1,\ldots,d}$, $k = 1, \ldots, p(\nu)$, sont les coefficients du modèle autorégressif à la saison ν, $\nu = 1, \ldots, s$. Le processus d'erreur $\boldsymbol{\epsilon} = \{\boldsymbol{\epsilon}_t, t \in \mathbb{Z}\}$ ci-dessus, $\boldsymbol{\epsilon}_t = (\epsilon_t(1), \ldots, \epsilon_t(d))^\top$, correspond à un bruit blanc périodique de moyenne zéro, c'est-à-dire $\boldsymbol{\epsilon}$ est composé de vecteurs aléatoires indépendants, tels que $E(\boldsymbol{\epsilon}_t) = \mathbf{0}$ et $E(\boldsymbol{\epsilon}_{ns+\nu}\boldsymbol{\epsilon}_{ns+\nu}^\top) = \mathbf{\Sigma}_{\boldsymbol{\epsilon}}(\nu)$, où la matrice de covariance $\mathbf{\Sigma}_{\boldsymbol{\epsilon}}(\nu) = (\boldsymbol{\sigma}_{\boldsymbol{\epsilon},ij}(\nu))_{i,j=1,\ldots,d}$ est considérée non singulière, $\nu = 1, \ldots, s$. Le processus PVAR est supposé avoir une moyenne égale à zéro $E(\mathbf{Y}_t) = \mathbf{0}$. En pratique, la tendance et les moyennes saisonnières sont enlevées de la série, c'est-à-dire le modèle à examiner est $\mathbf{Y}_{ns+\nu} - \boldsymbol{\mu}_\nu$, où en général l'espérance mathématique $E(\mathbf{Y}_{ns+\nu}) = \boldsymbol{\mu}_\nu$ peut être fonction de la saison ν. Si $s = 1$, alors le modèle PVAR défini ci-dessus se réduit au modèle autorégressif vectoriel (VAR).

Tout modèle PVAR défini dans la relation (1.2) peut être exprimé comme un modèle VAR en utilisant ce qu'on appelle le processus vectoriel empilé, dont les éléments sont les variables saisonnières pour toutes les saisons s. Par exemple,

pour un processus périodique $\mathbf{Y}_{ns+\nu}$ le vecteur empilé est donné par :

$$\mathbf{Y}_n^* = (\mathbf{Y}_{ns+s}^\top, \mathbf{Y}_{ns+s-1}^\top, \ldots, \mathbf{Y}_{ns+1}^\top)^\top.$$

L'étude des séries chronologiques de types ARMA présume généralement la stationnarité. Les processus périodiques sont non-stationnaires, mais un concept important est la stationnarité périodique, c'est-à-dire que la moyenne et la fonction d'autocovariance sont stationnaires dans un sens périodique. En utilisant l'équivalence algébrique entre la stationnarité multivariée et la corrélation périodique (voir Gladyshev (1961) ou Ula (1990)), le processus $\{\mathbf{Y}_n^*\}$ de dimension ds est stationnaire si et seulement si le processus $\{\mathbf{Y}_t\}$ de dimension d est périodiquement stationnaire de période s, dans le sens que :

$$\text{cov}(\mathbf{Y}_{n+s}, \mathbf{Y}_{m+s}) = \text{cov}(\mathbf{Y}_n, \mathbf{Y}_m),$$

pour tout entier n et m. Les processus périodiquement stationnaires sont aussi appelés cyclostationnaires (Lund et Basawa (2000)). La fonction d'autocovariance du processus $\{\mathbf{Y}_t\}$ de moyenne zéro est définie comme :

$$\mathbf{\Gamma}_\mathbf{Y}(h;\nu) = \text{cov}(\mathbf{Y}_{ns+\nu}, \mathbf{Y}_{ns+\nu-h}) = E(\mathbf{Y}_{ns+\nu}\mathbf{Y}_{ns+\nu-h}^\top),$$

qui dépendent à la fois du délai h et de la saison ν, mais pas de l'année n. L'autocovariance $\mathbf{\Gamma}_\mathbf{Y}(h;\nu)$ est interprétée comme périodique en ν de période s, en utilisant les relations :

$$\begin{aligned}
\mathbf{\Gamma}_\mathbf{Y}(h;\nu) &= \text{cov}(\mathbf{Y}_{ns+\nu}, \mathbf{Y}_{ns+\nu-h}), \\
&= \text{cov}(\mathbf{Y}_{(n+1)s+\nu}, \mathbf{Y}_{(n+1)s+\nu-h}), \\
&= \mathbf{\Gamma}_\mathbf{Y}(h;\nu+s).
\end{aligned}$$

Des arguments analogues nous mènent à la relation suivante pour les délais négatifs :

$$\mathbf{\Gamma}_\mathbf{Y}(-h;\nu) = \mathbf{\Gamma}_\mathbf{Y}^\top(h;\nu+h). \tag{1.3}$$

Pour les modèles de séries chronologiques périodiques multivariées, Ula (1990) a étudié les conditions de stationnarité (dans le sens périodique) pour les processus périodiques ARMA multivariés. Franses et Paap (2004) ont trouvé les conditions de stationnarité pour un modèle autorégressif périodique vectoriel (PVAR) avec

quatre saisons et un ordre autorégressif égal à un pour chaque saison, et ils ont étudié l'estimation des paramètres de ce modèle.

La modélisation des séries chronologiques implique généralement trois étapes principales : l'identification du modèle, l'estimation des paramètres et la validation du modèle avec des statistiques de test. La première étape a comme but d'identifier l'ordre des modèles périodiques. Un modèle saisonnier ARMA a une fonction d'autocovariance qui n'est fonction que du délai. Cependant, un modèle périodique possède une fonction d'autocovariance, définie dans un sens périodique, fonction du délai mais aussi de la saison. Ainsi, la fonction d'autocovariance ordinaire ne peut être utilisée pour l'identification des modèles périodiques. Anderson et Vecchia (1993) ont obtenu des résultats asymptotiques pour la fonction d'autocorrélation périodique des modèles univariés PARMA. Ula et Smadi (2003) ont utilisé les propriétés de la fonction d'autocorrélation périodique pour l'identification des ordres de modèles périodiques MA (PMA), qui est une extension de la technique d'identification de Box-Jenkins (Box et Jenkins (1970)) appliquée aux modèles MA et AR. Le critère d'information d'Akaike (Akaike (1974)) peut aussi être utilisé. Une des techniques utilisées présentement est de factoriser le critère bayésien d'information (BIC) pour obtenir un critère distinct pour chaque période (voir McLeod (1994)).

Compte tenu du caractère saisonnier des paramètres, l'estimation des processus périodiques est plus compliquée que celle des modèles autorégressifs et moyennes mobiles classiques. Pagano (1978) a étudié l'estimation des paramètres par la méthode des moments dans les modèles autorégressifs périodiques (PAR). Vecchia (1985b) a proposé un algorithme de maximum de vraisemblance pour l'estimation des modèles ARMA périodiques. Li (1988) a mis au point un algorithme de maximum de vraisemblance exact pour les modèles moyennes mobiles périodiques. Anderson, Meerschaert et Vecchia (1999) ont mis au point l'algorithme des innovations pour l'estimation de paramètres du modèle PARMA. Lütkepohl (2005) a étudié l'estimation par maximum de vraisemblance des paramètres du modèle général d'un processus stochastique PVAR, et il a considéré des tests statistiques permettant de tester l'invariance des coefficients du modèle.

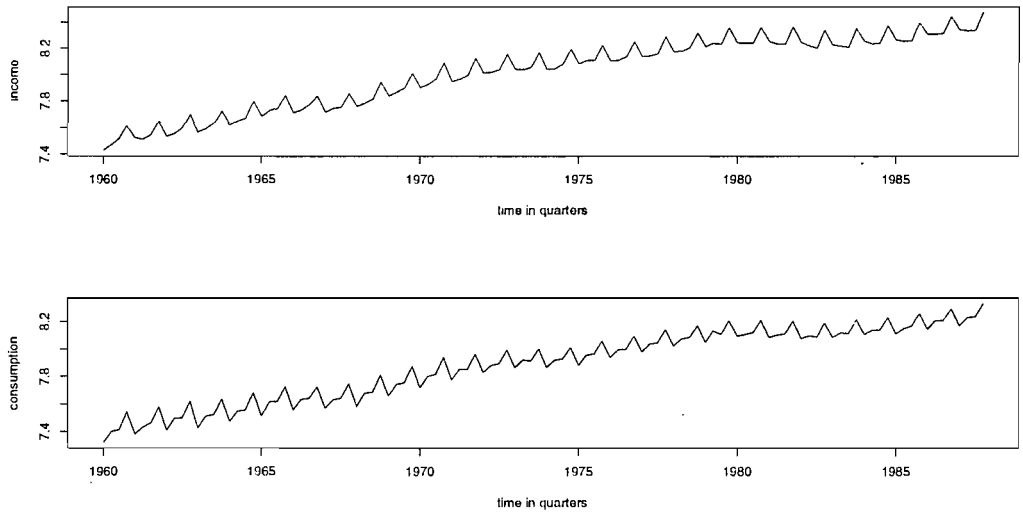
La validation du modèle, qui est la dernière étape du développement de la modélisation des séries chronologiques, est effectuée en utilisant les autocovariances et autocorrélations résiduelles. Les propriétés des autocovariances et des autocorrélations résiduelles ont été étudiées pour plusieurs modèles de séries chronologiques dans la monographie de Li (2004). Des tests portemanteaux pour les modèles périodiques univariés PAR ont été proposés par McLeod (1994) et Hipel et McLeod (1994).

Les séries chronologiques saisonnières et périodiques sont habituellement de nature très différente, et Lund et Basawa (1999) présentent une intéressante comparaison entre elles. La différence entre un modèle SAR (saisonnier autorégressif) et un modèle PAR (périodique autorégressif) peut être vue intuitivement en considérant une série économique mensuelle. Vu comme un modèle s'interprétant comme un modèle linéaire de régression, un modèle PAR d'ordre deux va utiliser les deux plus récentes observations (disons janvier et février) pour prédire l'observation du mois suivant (mars). Cependant, un modèle SAR d'ordre deux va utiliser les observations des mois de mars des deux plus récentes années pour prédire le mois de mars à venir. Ainsi, le modèle SAR va ignorer les deux plus récentes observations, qui sont potentiellement importantes.

Un exemple de série saisonnière est représenté dans la figure 1.1. Cette série correspond à des données trimestrielles sur le revenu et la consommation dans l'Allemagne de l'Ouest entre 1960 et 1987, données tirées de Lütkepohl (2005). Les données ont été transformées en utilisant le logarithme. Une tendance croissante peut être observée, avec une variation saisonnière dominante. Cette variation saisonnière est identifiée par la présence de cycles répétitifs. Le graphique 1.1 n'indique pas si cette variation saisonnière est importante ou si la variation saisonnière semble constante dans le temps.

Pour ces raisons, on considère également les graphiques 1.2 et 1.3. Ces graphiques contiennent quatre lignes, dont chacune est associée avec un des quatre trimestres. Les données utilisées ici ont été différenciées, plus précisément en utilisant la première différence $1 - B$ des données transformées par la transformation logarithmique. D'autres méthodes de différenciation ont été utilisées pour des

FIGURE 1.1. Les données sur le revenu (en haut) et la consommation (en bas) pour l'Allemagne de l'Ouest entre les années 1960-1987. Les données ont été transformées en appliquant le logarithme pour chaque variable.



données périodiques dans le chapitre 4. Pour plus de détails, voir Franses et Paap (2004, Chapitre 4). Les graphiques 1.2 et 1.3 nous donnent une première appréciation de la variation saisonnière. Par exemple, si les quatre lignes sont distinctes et relativement éloignées les unes des autres on a une bonne indication d'une variation saisonnière importante. Si les lignes se croisent alors on déduit que la saisonnalité change dans le temps (par exemple, le phénomène estival semble se déplacer plus tardivement, comme durant la saison hivernale). Ce type de graphique a été introduit par Franses (1994).

La périodicité dans la moyenne peut être observée facilement en utilisant des graphiques similaires à ceux représentés dans la figure 1.1. Toutefois, la périodicité dans les moments d'ordre supérieur n'est pas si facile à observer. Afin de différencier les modèles périodiques des modèles non périodiques, des tests de type maximum de vraisemblance peuvent vérifier si les coefficients semblent constants d'une période à l'autre. Voir par exemple Lütkepohl (2005). McLeod (1993) propose un test diagnostique basé sur les autocorrélations résiduelles de délai un afin

de détecter la périodicité dans les modèles saisonniers ARMA. Il est important de préciser que les modèles périodiques restent périodiques même après l'application d'un filtre (saisonnier ou non).

FIGURE 1.2. Taux de croissance trimestriel de l'Allemagne de l'Ouest pour le revenu, 1960-1987 (avec transformation logarithmique).

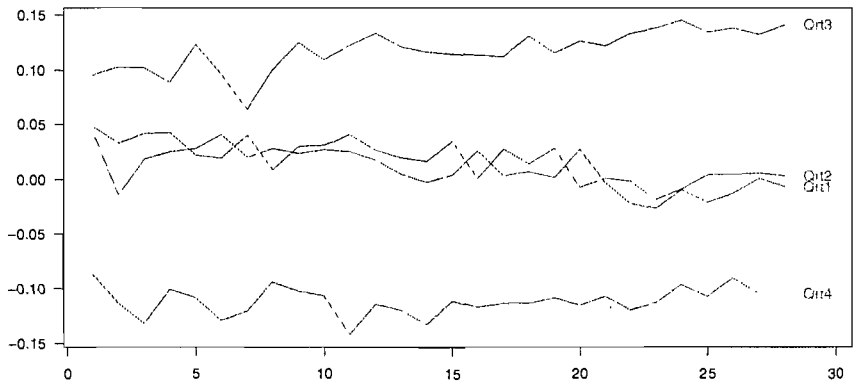
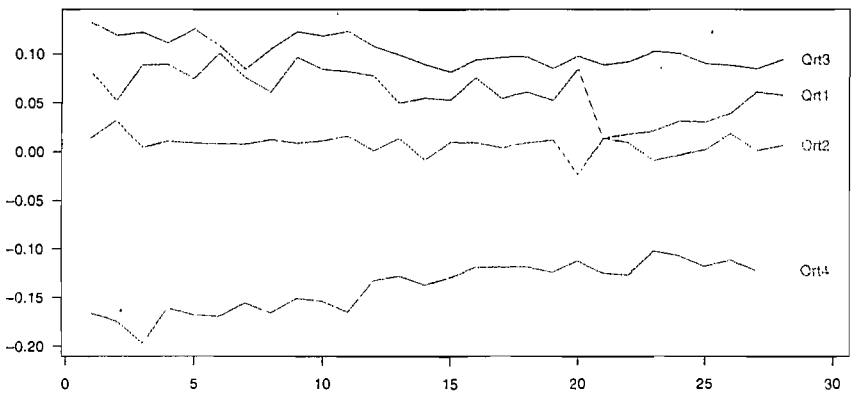


FIGURE 1.3. Taux de croissance trimestriel de l'Allemagne de l'Ouest pour la consommation, 1960-1987 (avec transformation logarithmique).



Dans la section suivante, nous donnons les objectifs spécifiques de notre recherche.

1.1. Objectifs de la recherche

L'objectif principal de la recherche présentée ici est de fournir des outils statistiques afin de décrire des séries chronologiques périodiques ou saisonnières. L'estimation est considérée et une attention particulière porte sur la validation des modèles avec des statistiques de test de type diagnostic.

Objectifs spécifiques :

(a) Les propriétés asymptotiques des estimateurs par moindres carrés des paramètres du modèle SVARMA sont considérées, et de nouveaux résultats sont obtenus. Afin d'obtenir des représentations plus parcimonieuses pour les séries chronologiques vectorielles, nous discutons aussi l'estimation avec des contraintes linéaires imposées sur les paramètres du modèle. Un deuxième objectif est d'obtenir la distribution asymptotique des matrices d'autocovariances résiduelles pour les modèles SVARMA. Ces résultats généralisent un théorème de McLeod (1978), qui est valable pour les modèles univariés saisonniers. Comme application de ces résultats, un test portemanteau est dégagé et nous étudions sa distribution asymptotique, qui est approximativement khi-carré.

(b) Dans le second projet, nous obtenons dans un premier temps de nouveaux résultats sur les estimateurs des moindres carrés des paramètres du modèle PVAR défini par (1.2). Comme les processus multivariés périodiques impliquent un nombre important de paramètres indépendants, nous considérons les situations où il existe des contraintes linéaires sur les paramètres d'une saison. Des cas particuliers importants sont les modèles où certains paramètres sont contraints à zéro. Comme second objectif, nous trouvons la distribution asymptotique des matrices d'autocovariances et d'autocorrélations résiduelles dans le cadre de modèles PVAR. Nos résultats généralisent la littérature dans plusieurs directions. En outre, nos résultats asymptotiques généralisent des théorèmes obtenus par McLeod (1994) pour les modèles PAR. Comme application de ces résultats, un test portemanteau basé sur un nombre fixé de matrices d'autocovariances résiduelles est proposé afin de diagnostiquer des modèles PVAR. Nous discutons la distribution

asymptotique de ces nouveaux tests statistiques et nous considérons la version modifiée de ces tests ayant de meilleures propriétés pour des tailles échantillonales modérément petites.

(c) Dans le dernier chapitre, nous introduisons un modèle multivarié saisonnier autorégressif avec des paramètres périodiques, abrégé par le sigle SPVAR, qui comprend le modèle autorégressif multivarié saisonnier ainsi que le modèle autorégressif périodique multivarié comme cas particuliers. La combinaison de modèles périodiques et saisonniers a été examinée par Basawa, Lund et Shao (2004) dans la situation particulière d'un processus de premier ordre. Ils ont étudié les conditions de stationnarité (dans le sens périodique) et la distribution asymptotique des estimateurs des moindres carrés. Ici, nous étendons les travaux de Basawa *et al.* (2004) dans trois directions. Tout d'abord, nous généralisons le processus du premier ordre à un processus saisonnier autorégressif avec des paramètres périodiques d'ordres p_1 et p_2, où p_1 et p_2 désignent les ordres saisonnier et autorégressif, respectivement. Nous étudions les conditions de stationnarité périodique dans cette classe de processus stochastiques et nous trouvons explicitement les fonctions d'autocovariances théoriques associées. Deuxièmement, nous présentons des résultats asymptotiques des estimateurs des moindres carrés des paramètres du modèle pour le modèle SPVAR. Troisièmement, nous considérons le test diagnostic pour les modèles SPVAR. La validation des modèles SPVAR n'a pas été étudiée dans Basawa *et al.* (2004). Afin de diagnostiquer les modèles SPVAR, nous obtenons la distribution asymptotique des matrices d'autocovariances et d'autocorrélations résiduelles dans cette classe de modèles. Comme application utile de ces résultats asymptotiques, un test portemanteau basé sur un nombre fixé de matrices résiduelles est proposé afin de diagnostiquer les modèles SPVAR.

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Chapitre 2

ON MULTIPLICATIVE SEASONAL MODELLING FOR VECTOR TIME SERIES

Cet article a été soumis pour publication en décembre 2008 dans une revue avec comité de lecture. Le premier auteur est Eugen Ursu et le coauteur est le directeur de recherche Pierre Duchesne.

Abstract : Many time series encountered in real applications display seasonal behavior. In this paper, we consider multiplicative seasonal vectorial autoregressive moving average (SVARMA) models to describe seasonal vector time series. We discuss least squares estimation of the model parameters, allowing them to satisfy general linear constraints. Having fitted a model, residual autocovariances (or autocorrelations) have been found useful in checking time series models. Consequently, we obtain the asymptotic distributions of the residual autocovariance matrices. As applications of these results, portmanteau test statistics are proposed and their asymptotic distributions are studied. The finite-sample properties of the portmanteau test statistics are evaluated using Monte Carlo experiments.
Key words and phrases : Diagnostic checking ; seasonal time series ; portmanteau test statistics ; residual autocovariance matrices ; vector time series.
Mathematics subject classification codes (2000) : primary 62M10 ; secondary 62H10.

2.1. INTRODUCTION

Seasonality naturally occurs in many time series coming from various fields of study, such as meteorology, hydrology or economics, amongst others. In univariate time series, the multiplicative seasonal moving average (SARMA) time series model is widely used in the modelling of seasonal time series (see, e.g., Hipel and McLeod (1994, Chapter 12)). A famous example is the so-called airline model (see, e.g., Box and Jenkins (1970) and Li (2004)) which has been used originally to model monthly totals of international airline passengers. To describe multivariate time series data, multiplicative seasonal vectorial autoregressive moving average (SVARMA) models can be used. That class of time series models has been introduced by Reinsel (1997), and is composed of multivariate generalizations of SARMA models. Let $\mathbf{Y} = \{\mathbf{Y}_t, t \in \mathbb{Z}\}$ be a stochastic process, where $\mathbf{Y}_t = (Y_t(1), \ldots, Y_t(d))^\top$ corresponds to a random vector of dimension d. Let B and B^s be the usual and seasonal lag operators, respectively, where s represents a predetermined value corresponding to the seasonal period. The stochastic process $\mathbf{Y}$ can be written as a multiplicative seasonal autoregressive moving average process, denoted SVARMA$(p,q) \times (P,Q)_s$, when it satisfies the linear difference equation :

$$\mathbf{\Lambda}(B^s)\mathbf{\Phi}(B)\mathbf{Y}_t = \mathbf{\Xi}(B^s)\mathbf{\Theta}(B)\boldsymbol{\epsilon}_t, \ t \in \mathbb{Z}, \tag{2.1}$$

where the nonseasonal AR and MA operators are defined by

$$\mathbf{\Phi}(B) = \mathbf{I}_d - \mathbf{\Phi}_1 B - \ldots - \mathbf{\Phi}_p B^p \quad \text{and} \quad \mathbf{\Theta}(B) = \mathbf{I}_d - \mathbf{\Theta}_1 B - \ldots - \mathbf{\Theta}_q B^q,$$

respectively, while the seasonal AR and MA operators are given by

$$\mathbf{\Lambda}(B^s) = \mathbf{I}_d - \mathbf{\Lambda}_1 B^s - \ldots - \mathbf{\Lambda}_P B^{sP} \quad \text{and} \quad \mathbf{\Xi}(B^s) = \mathbf{I}_d - \mathbf{\Xi}_1 B^s - \ldots - \mathbf{\Xi}_Q B^{sQ},$$

respectively. Here, the $d \times d$ matrix $\mathbf{I}_d$ denotes the identity matrix of order d. The random vector $\mathbf{Y}_t$ in (2.1) is supposed to have a zero mean. In practical applications, trends and seasonal means are first removed from the time series. The error process $\boldsymbol{\epsilon} = \{\boldsymbol{\epsilon}_t, t \in \mathbb{Z}\}$, $\boldsymbol{\epsilon}_t = (\epsilon_t(1), \ldots, \epsilon_t(d))^\top$, of the SVARMA model corresponds to a zero mean white noise, that is $\boldsymbol{\epsilon}$ is composed of uncorrelated

random vectors, such that $E(\boldsymbol{\epsilon}_t) = \mathbf{0}$ and $E(\boldsymbol{\epsilon}_t \boldsymbol{\epsilon}_t^\top) = \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}$, where the error covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}} = (\sigma_{\boldsymbol{\epsilon},ij})_{i,j=1,\ldots,d}$ is assumed to be non singular. The seasonal and nonseasonal AR and MA operators are $d \times d$ matrices. In general, they do not commute and the order in which these matrices intervene in (2.1) will make a difference. See also Reinsel (1997) who study special cases of (2.1). We assume that the SVARMA stochastic process is causal and invertible. From the results for VARMA stochastic processes, the causality conditions are $\det\{\boldsymbol{\Phi}(z)\} \neq 0$ and $\det\{\boldsymbol{\Lambda}(z^s)\} \neq 0$ for all complex numbers z satisfying the condition $|z| \leq 1$, where $\det(\mathbf{A})$ stands for the determinant of the square matrix $\mathbf{A}$. Similarly, the stochastic process $\{\mathbf{Y}_t\}$ is invertible if and only if $\det\{\boldsymbol{\Theta}(z)\} \neq 0$ and $\det\{\boldsymbol{\Xi}(z^s)\} \neq 0$ for $|z| \leq 1$. These requirements represent the natural extensions to seasonal vector time series of the causality and invertibility conditions in univariate seasonal processes (see, e.g., Brockwell and Davis (1991, p. 323) or Hipel and McLeod (1994, p. 423)). The model parameters are supposed to be such that the SVARMA structure is simply identified (see, e.g., Dunsmuir and Hannan (1976), Deistler, Dunsmuir and Hannan (1978), Poskitt and Tremayne (1982), Reinsel (1997) and Lütkepohl (2005, Section 12.1)).

Periodic vector time series provide an alternative approach to describe seasonal time series. These models are non-stationary and they are designed to model time series data which display periodic statistical structure (for vector time series, see, e.g., Franses and Paap (2004), Lütkepohl (2005) and Ursu and Duchesne (2007, 2008)). Seasonal and periodic time series models are quite different, and Lund and Basawa (1999) present an interesting comparison between them. It is usually recognized that periodic time series models rely typically on a large number of parameters. For example, a periodic VAR(1) model for bivariate monthly data involves 48 independent parameters, and to find parsimonious representation is not always an easy task (see, e.g., Ursu and Duchesne (2008)). On the other hand, the multivariate version of the airline data applied to the time series data $\mathbf{W}_t = (1 - B)(1 - B^{12})\mathbf{Y}_t$ is the SVARMA$(0, 1) \times (0, 1)_{12}$ model, which implies for bivariate monthly data a total of eight parameters. The class of SVARMA models is expected to complement advantageously periodic models in practical

applications, particularly for seasonal vector time series with small to moderate sample sizes.

As a first objective of this paper, the asymptotic properties of the least squares estimators of the model parameters in the SVARMA model (2.1) are considered, and new results are given. In order to obtain parsimonious representations of vector time series data, we discuss estimation under linear constraints imposed on the model parameters. From a model-building point of view, it appears desirable to have diagnostic tests to check the adequacy of a particular fitted model. Diagnostic test statistics for VARMA models have been studied by many authors (see, e.g., Hosking (1980), Li and McLeod (1981), Poskitt and Tremayne (1982), Li (2004) and Lütkepohl (2005)). Consequently, a second objective is to derive the asymptotic distributions of the residual autocovariance matrices in the context of SVARMA models. These results generalize a theorem of McLeod (1978) which is valid for univariate seasonal models. As applications of our results, portmanteau test statistics are considered and we study their asymptotic distributions, which are approximatively chi-square.

The paper is organized as follows. In Section 2.2, some preliminaries are given. Least squares estimators are studied in Section 2.3. In Section 2.4, the asymptotic distributions of the residual autocovariance matrices under the null hypothesis of model adequacy are derived. We describe applications for diagnostic checking based on these asymptotic results, by considering portmanteau test statistics. In Section 2.5, Monte Carlo experiments are conducted. Throughout the paper, $||\mathbf{A}|| = \{\mathrm{tr}(\mathbf{A}\mathbf{A}^\top)\}^{1/2}$ denotes the Euclidian norm of the matrix $\mathbf{A}$, where $\mathrm{tr}(\mathbf{B})$ corresponds to the trace of the square matrix $\mathbf{B}$. The notation $\mathrm{vec}(\mathbf{A})$ represents the vector obtained by stacking the columns of $\mathbf{A}$ and '$\otimes$' is the Kronecker product (see, e.g., Harville (1997)). The symbols '$\xrightarrow{d}$' and '$\xrightarrow{p}$' stand for convergence in distribution and probability, respectively. The d-dimensional normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ is noted $\mathcal{N}_d(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. The canonical basis of $\mathbb{R}^d$ is given by $\{\mathbf{e}_i\}_{i=1,\ldots,d}$, where $\mathbf{e}_i$ is a vector with one in position i, and zero elsewhere. Similarly, the canonical basis of $\mathbb{R}^{d\times d}$ is $\{\mathbf{E}_{ij}\}_{i,j=1,\ldots,d}$, where $\mathbf{E}_{ij}$ is a $d \times d$ matrix with one in position (i, j), and zero elsewhere. All the asymptotic

results are taken as $N \to \infty$, where N denotes the sample size of the vector time series.

2.2. PRELIMINARIES

This section provides the basic properties of SVARMA stochastic processes. If $\mathbf{Y} = \{\mathbf{Y}_t, t \in \mathbb{Z}\}$ generated by (2.1) corresponds to a causal stochastic process, it can be represented through an infinite order moving average expansion :

$$\mathbf{Y}_t = \sum_{k=0}^{\infty} \mathbf{\Psi}_k \boldsymbol{\epsilon}_{t-k}, \tag{2.2}$$

where $\mathbf{\Psi}_0 = \mathbf{I}_d$ and $\mathbf{\Psi}_k = \mathbf{0}$ if $k < 0$. The sequence $\{\mathbf{\Psi}_k\}_{k=0,1,\ldots}$ composed of the $d \times d$ matrices $\mathbf{\Psi}_k$ is presumed summable, in the sense that $\sum_{k=0}^{\infty} ||\mathbf{\Psi}_k|| < \infty$.

The theoretical lag h autocovariance matrix of the zero-mean stochastic process $\{\mathbf{Y}_t\}$ is defined as $\mathbf{\Gamma}_{\mathbf{Y}}(h) = \text{cov}(\mathbf{Y}_t, \mathbf{Y}_{t-h}) = E(\mathbf{Y}_t \mathbf{Y}_{t-h}^{\top})$. Using the moving average expression (2.2), it follows that its theoretical autocovariance function satisfies

$$\mathbf{\Gamma}_{\mathbf{Y}}(h) = \sum_{k=0}^{\infty} \mathbf{\Psi}_k \mathbf{\Sigma}_{\boldsymbol{\epsilon}} \mathbf{\Psi}_{k-h}^{\top}, \quad h = 0, 1, \ldots,$$

and

$$\mathbf{\Gamma}_{\mathbf{Y}}(-h) = \mathbf{\Gamma}_{\mathbf{Y}}^{\top}(h) \quad \text{if} \quad h = -1, -2, \ldots.$$

We introduce the $(d^2p) \times 1$ vector

$$\boldsymbol{\phi} = (\text{vec}^{\top}(\mathbf{\Phi}_1), \ldots, \text{vec}^{\top}(\mathbf{\Phi}_p))^{\top},$$

the $(d^2q) \times 1$ vector

$$\boldsymbol{\theta} = (\text{vec}^{\top}(\mathbf{\Theta}_1), \ldots, \text{vec}^{\top}(\mathbf{\Theta}_q))^{\top},$$

whose elements are the model parameters associated to the nonseasonal operators, the $(d^2P) \times 1$ vector

$$\boldsymbol{\lambda} = (\text{vec}^{\top}(\mathbf{\Lambda}_1), \ldots, \text{vec}^{\top}(\mathbf{\Lambda}_P))^{\top}$$

and the $(d^2Q) \times 1$ vector

$$\boldsymbol{\xi} = (\text{vec}^\top(\boldsymbol{\Xi}_1), \ldots, \text{vec}^\top(\boldsymbol{\Xi}_Q))^\top$$

whose elements correspond to the model parameters of the seasonal operators. We collect all the parameters of the SVARMA model in the $\{d^2(p+q+P+Q)\} \times 1$ vector $\boldsymbol{\beta} = (\boldsymbol{\beta}_1^\top, \boldsymbol{\beta}_2^\top, \boldsymbol{\beta}_3^\top, \boldsymbol{\beta}_4^\top)^\top$, where $\boldsymbol{\beta}_1 = \boldsymbol{\phi}$, $\boldsymbol{\beta}_2 = \boldsymbol{\theta}$, $\boldsymbol{\beta}_3 = \boldsymbol{\lambda}$ and $\boldsymbol{\beta}_4 = \boldsymbol{\xi}$. We assume that for a known $\{d^2(p+q+P+Q)\} \times K$ matrix $\mathbf{R}$ of rank K, and a known $\{d^2(p+q+P+Q)\} \times 1$ vector $\mathbf{b}$, the vectors $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are related via the following linear transformation :

$$\boldsymbol{\beta} = \mathbf{R}\boldsymbol{\gamma} + \mathbf{b}. \tag{2.3}$$

The elements of the $K \times 1$ vector $\boldsymbol{\gamma}$ correspond to the unknown parameters. If $\mathbf{R} = \mathbf{I}_{d^2(p+q+P+Q)}$, $\mathbf{b} = \mathbf{0}$, we obtain the full unconstrained situation. More generally, $\mathbf{R}$ and $\mathbf{b}$ allow us to specify linear constraints between the parameters. That general specification includes the important special case of zero constraints on the parameters, which is useful in finding parsimonious models.

For any particular vector $\dot{\boldsymbol{\beta}} = (\dot{\boldsymbol{\beta}}_1^\top, \dot{\boldsymbol{\beta}}_2^\top, \dot{\boldsymbol{\beta}}_3^\top, \dot{\boldsymbol{\beta}}_4^\top)^\top$, the model residuals are defined by $\dot{\boldsymbol{\epsilon}}_t = \dot{\boldsymbol{\Theta}}^{-1}(B)\dot{\boldsymbol{\Xi}}^{-1}(B^s)\dot{\boldsymbol{\Lambda}}(B^s)\dot{\boldsymbol{\Phi}}(B)\mathbf{Y}_t$. Let $\boldsymbol{\Gamma}_{\boldsymbol{\epsilon}}(h) = \text{cov}(\boldsymbol{\epsilon}_t, \boldsymbol{\epsilon}_{t-h})$ be the lag h theoretical autocovariance matrix of the error process $\boldsymbol{\epsilon}$. We introduce the sample autocovariance matrices $\mathbf{C}_{\dot{\boldsymbol{\epsilon}}}(h) = \left(C_{\dot{\boldsymbol{\epsilon}},ij}(h)\right)_{i,j=1,\ldots,d}$:

$$\mathbf{C}_{\dot{\boldsymbol{\epsilon}}}(h) = \begin{cases} N^{-1}\sum_{t=h+1}^{N} \dot{\boldsymbol{\epsilon}}_t \dot{\boldsymbol{\epsilon}}_{t-h}^\top, & h \geq 0, \\ \mathbf{C}_{\dot{\boldsymbol{\epsilon}}}^\top(-h), & h < 0. \end{cases}$$

Let $\mathbf{c}_{\dot{\boldsymbol{\epsilon}}}(h) = \text{vec}\{\mathbf{C}_{\dot{\boldsymbol{\epsilon}}}(h)\}$. The vector of sample autocovariances are collected in the random vector $\mathbf{c}_{\dot{\boldsymbol{\epsilon}}} = (\mathbf{c}_{\dot{\boldsymbol{\epsilon}}}^\top(1), \ldots, \mathbf{c}_{\dot{\boldsymbol{\epsilon}}}^\top(M))^\top$, where the maximal lag order M represents a fixed integer with respect to N, with $1 \leq M < N$. In the next section, the asymptotic properties of the least squares estimators from a causal and invertible SVARMA model are discussed.

2.3. ASYMPTOTIC PROPERTIES OF THE LEAST SQUARES ESTIMATORS

2.3.1. Full unconstrained case

First, we study the asymptotic properties when no linear constraints exist on the model parameters. Consider the time series data $\mathbf{Y}_t$, $t = 1, 2, \ldots, N$. The least squares estimators of $\boldsymbol{\beta}$ are obtained by minimizing the least squares criterion $S_{LS} \equiv S_{LS}(\boldsymbol{\beta}; \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}) = \sum_{t=1}^{N} \boldsymbol{\epsilon}_t^\top \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1} \boldsymbol{\epsilon}_t$. Using classical results on matrix differentiation (see, e.g., Harville (1997, Chapter 15)), differentiating S_{LS} with respect to $\boldsymbol{\beta}$ allow us to show that the least squares estimators, denoted $\hat{\boldsymbol{\beta}}$, satisfy the following system :

$$\frac{\partial S_{LS}}{\partial \boldsymbol{\beta}^\top} = 2 \sum_{t=1}^{N} \boldsymbol{\epsilon}_t^\top \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1} \frac{\partial \boldsymbol{\epsilon}_t}{\partial \boldsymbol{\beta}^\top} = \mathbf{0}^\top. \tag{2.4}$$

Due to the presence of moving average components, it is useful to introduce the following polynomials : $\mathcal{G}_1(B) = \boldsymbol{\Theta}^{-1}(B)\boldsymbol{\Xi}^{-1}(B^s)\boldsymbol{\Lambda}(B^s)$, $\mathcal{G}_2(B) = \boldsymbol{\Theta}^{-1}(B)\boldsymbol{\Xi}^{-1}(B^s)$ and $\mathcal{G}_3(B) = \boldsymbol{\Theta}^{-1}(B)$. In general, these expressions are infinite power series in the lag operator B. Thus $\mathcal{G}_i(B) = \sum_{l=0}^{\infty} \mathbf{G}_{il} B^l$, $i \in \{1, 2, 3\}$. Consequently,

$$\boldsymbol{\epsilon}_t = \mathcal{G}_1(B)\boldsymbol{\Phi}(B)\mathbf{Y}_t = \sum_{l=0}^{\infty} \mathbf{G}_{1l}\mathbf{Y}_{t-l} - \sum_{i=1}^{p}\sum_{l=0}^{\infty} \mathbf{G}_{1l}\boldsymbol{\Phi}_i \mathbf{Y}_{t-l-i}. \tag{2.5}$$

Vectorizing (2.5) and using $\text{vec}(\mathbf{ABC}) = (\mathbf{C}^\top \otimes \mathbf{A})\text{vec}(\mathbf{B})$, the derivative $\partial \boldsymbol{\epsilon}_t / \partial \text{vec}^\top(\boldsymbol{\Phi}_i)$ is :

$$\frac{\partial \boldsymbol{\epsilon}_t}{\partial \text{vec}^\top(\boldsymbol{\Phi}_i)} = -\sum_{l=0}^{\infty} (\mathbf{Y}_{t-l-i}^\top \otimes \mathbf{G}_{1l}), \; i = 1, \ldots, p. \tag{2.6}$$

Let $\boldsymbol{\zeta}_k(B) = (B, B^2, \ldots, B^k)^\top$ be a $k \times 1$ vector containing the powers of the backward shift operator B. The derivatives in (2.6) can be collected in the following $d \times (d^2 p)$ matrix :

$$\frac{\partial \boldsymbol{\epsilon}_t}{\partial \boldsymbol{\phi}^\top} = -\sum_{l=0}^{\infty} \{\boldsymbol{\zeta}_p^\top(B) \otimes \mathbf{Y}_{t-l}^\top \otimes \mathbf{G}_{1l}\} = -\{\boldsymbol{\zeta}_p^\top(B) \otimes \mathbf{Y}_t^\top \otimes \mathcal{G}_1(B)\}, \tag{2.7}$$

adopting the convention that the backward shift operator B is applied to the preceding component in the Kronecker product intervening in expression (2.7).

See Poskitt and Tremayne (1982) for the same convention (note that this formulation is used only for notational convenience). Let $\boldsymbol{\epsilon}_t = \mathcal{G}_2(B)\boldsymbol{\Lambda}(B^s)\mathbf{W}_t$, where $\mathbf{W}_t = \boldsymbol{\Phi}(B)\mathbf{Y}_t$. Similar arguments give the $d \times (d^2P)$ matrix :

$$\frac{\partial \boldsymbol{\epsilon}_t}{\partial \boldsymbol{\lambda}^\top} = -\sum_{l=0}^{\infty}\{\zeta_P^\top(B^s) \otimes \mathbf{W}_{t-l}^\top \otimes \mathbf{G}_{2l}\} = -\{\zeta_P^\top(B^s) \otimes \mathbf{W}_t^\top \otimes \mathcal{G}_2(B)\}.$$

In order to compute the derivatives with respect to the moving-average components, consider the auxiliary model $\mathbf{V}_t = \boldsymbol{\Theta}(B)\boldsymbol{\epsilon}_t$, where

$$\mathbf{V}_t = \boldsymbol{\Xi}^{-1}(B^s)\boldsymbol{\Lambda}(B^s)\boldsymbol{\Phi}(B)\mathbf{Y}_t.$$

Using arguments similar to those of Li and Mcleod (1981, Section 4) and invoking $\partial \mathrm{vec}(\mathbf{AB})/\partial \boldsymbol{\beta}^\top = (\mathbf{I}_q \otimes \mathbf{A})\partial \mathrm{vec}(\mathbf{B})/\partial \boldsymbol{\beta}^\top + (\mathbf{B}^\top \otimes \mathbf{I}_n)\partial \mathrm{vec}(\mathbf{A})/\partial \boldsymbol{\beta}^\top$, for matrices $\mathbf{A}$ and $\mathbf{B}$ of dimensions $n \times p$ and $p \times q$, respectively, the componentwise derivatives are given by

$$\partial \boldsymbol{\epsilon}_t/\partial \theta_{\alpha\beta,i} = \boldsymbol{\Theta}^{-1}(B)\mathbf{E}_{\alpha\beta}\boldsymbol{\epsilon}_{t-i} = \boldsymbol{\Theta}^{-1}(B)\mathbf{e}_\alpha \epsilon_{t-i}(\beta),$$

where $\boldsymbol{\Theta}_i = (\theta_{\alpha\beta,i})_{\alpha,\beta=1\ldots,d}$. It follows that $\partial \boldsymbol{\epsilon}_t/\partial \mathrm{vec}^\top(\boldsymbol{\Theta}_i)$ is the $d \times d^2$ matrix :

$$\begin{aligned}\frac{\partial \boldsymbol{\epsilon}_t}{\partial \mathrm{vec}^\top(\boldsymbol{\Theta}_i)} &= \left(\boldsymbol{\Theta}^{-1}(B)\epsilon_{t-i}(1)\ \boldsymbol{\Theta}^{-1}(B)\epsilon_{t-i}(2)\cdots\boldsymbol{\Theta}^{-1}(B)\epsilon_{t-i}(d)\right),\ i = 1,\ldots,q,\\ &= \sum_{l=0}^{\infty}(\boldsymbol{\epsilon}_{t-l-i}^\top \otimes \mathbf{G}_{3l}),\ i = 1,\ldots,q.\end{aligned}$$

Thus, adopting the Poskitt and Tremayne (1982) convention gives the $d \times (d^2q)$ matrix :

$$\frac{\partial \boldsymbol{\epsilon}_t}{\partial \boldsymbol{\theta}^\top} = \sum_{l=0}^{\infty}\{\zeta_q^\top(B) \otimes \boldsymbol{\epsilon}_{t-l}^\top \otimes \mathbf{G}_{3l}\} = \{\zeta_q^\top(B) \otimes \boldsymbol{\epsilon}_t^\top \otimes \boldsymbol{\Theta}^{-1}(B)\}.$$

Let $\mathbf{v}_t = \boldsymbol{\Theta}(B)\boldsymbol{\epsilon}_t$. Similar calculations give the relations :

$$\frac{\partial \boldsymbol{\epsilon}_t}{\partial \boldsymbol{\xi}^\top} = \sum_{l=0}^{\infty}\{\zeta_Q^\top(B^s) \otimes \mathbf{v}_{t-l}^\top \otimes \mathbf{G}_{2l}\} = \{\zeta_Q^\top(B^s) \otimes \mathbf{v}_t^\top \otimes \mathcal{G}_2(B)\}.$$

Setting $P = Q = 0$, $\boldsymbol{\Lambda}(B) = \boldsymbol{\Xi}(B) = \mathbf{I}_d$, we retrieve the results of Poskitt and Tremayne (1982, p. 116) for VARMA models. We now introduce the $\{d^2(p+q+P+Q)\} \times 1$ random vector $\mathbf{Z}_t$, such that $\mathbf{Z}_t^\top = \boldsymbol{\epsilon}_t^\top \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\partial \boldsymbol{\epsilon}_t/\partial \boldsymbol{\beta}^\top)$. Lemma 2.1 is needed in order to establish the asymptotic behavior of the least squares estimators $\hat{\boldsymbol{\beta}}$ of $\boldsymbol{\beta}$.

Lemma 2.1. *Let a stochastic process* $\mathbf{Y} = \{\mathbf{Y}_t, t \in \mathbb{Z}\}$, $\mathbf{Y}_t = (Y_t(1), \ldots, Y_t(d))^\top$, *be generated by a causal and invertible SVARMA difference equation (2.1). Suppose that the error term* $\boldsymbol{\epsilon} = \{\boldsymbol{\epsilon}_t\}$ *corresponds to a white noise composed of independent random vectors satisfying* $E(\boldsymbol{\epsilon}_t) = \mathbf{0}$ *and* $\mathrm{var}(\boldsymbol{\epsilon}_t) = \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}$. *Assume that the fourth-order moments of* $\boldsymbol{\epsilon}_t$ *are finite :* $E\{|\epsilon_t(i)\epsilon_t(j)\epsilon_t(k)\epsilon_t(l)|\} < \infty$, $\forall i, j, k, l = 1, \ldots, d$, $\forall t \in \mathbb{Z}$. *Then the following asymptotic results hold :*

$$N^{-1}\sum_{t=1}^{N}\mathbf{Z}_t \xrightarrow{p} \mathbf{0},$$

$$N^{-1}\sum_{t=1}^{N}\mathbf{Z}_t\mathbf{Z}_t^\top \xrightarrow{p} \boldsymbol{\Omega}_{\boldsymbol{\beta}},$$

$$N^{-1/2}\sum_{t=1}^{N}\mathbf{Z}_t \xrightarrow{d} \mathcal{N}_{d^2(p+q+P+Q)}(\mathbf{0}, \boldsymbol{\Omega}_{\boldsymbol{\beta}}),$$

$$N^{-1}\partial(\sum_{t=1}^{N}\mathbf{Z}_t)/\partial\boldsymbol{\beta}^\top \xrightarrow{p} \boldsymbol{\Omega}_{\boldsymbol{\beta}},$$

where the matrix $\boldsymbol{\Omega}_{\boldsymbol{\beta}}$ *corresponds to the* $\{d^2(p+q+P+Q)\} \times \{d^2(p+q+P+Q)\}$ *matrix :*

$$\boldsymbol{\Omega}_{\boldsymbol{\beta}} = E\left(\frac{\partial\boldsymbol{\epsilon}_t^\top}{\partial\boldsymbol{\beta}}\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}\frac{\partial\boldsymbol{\epsilon}_t}{\partial\boldsymbol{\beta}^\top}\right). \tag{2.8}$$

PROOF. Let $\mathcal{F}_t = \sigma(\mathbf{Y}_t, \mathbf{Y}_{t-1}, \ldots)$ be the sigma-algebra associated with $\{\mathbf{Y}_{t-k}, k \geq 0\}$. Noting that $\{\mathbf{Z}_t, \mathcal{F}_{t-1}\}$ represents a martingale difference sequence, the proof of Lemma 2.1 follows using the law of large number and the central limit theorem for martingale difference sequences. For more detailed and similar arguments, see for example Ursu and Duchesne (2007) or Proposition 4.1, chapter 4. □

Theorem 2.1 states the asymptotic distribution of the least squares estimators $\hat{\boldsymbol{\beta}}$ of $\boldsymbol{\beta}$.

Theorem 2.1. *Let a stochastic process* $\mathbf{Y} = \{\mathbf{Y}_t, t \in \mathbb{Z}\}$, $\mathbf{Y}_t = (Y_t(1), \ldots, Y_t(d))^\top$, *be generated by a causal and invertible* $SVARMA(p, q) \times (P, Q)_s$ *time series model. Under the conditions of Lemma 2.1, the distributions of the least squares*

estimators $\hat{\boldsymbol{\beta}}$ of the model parameters $\boldsymbol{\beta}$ are asymptotically normal :

$$N^{1/2}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \xrightarrow{d} \mathcal{N}_{d^2(p+q+P+Q)}\left(\mathbf{0}, \boldsymbol{\Omega}_{\boldsymbol{\beta}}^{-1}\right). \tag{2.9}$$

The matrix $\boldsymbol{\Omega}_{\boldsymbol{\beta}}$ is defined by 2.8.

PROOF. Using arguments similar to those found in Fuller (1996, Theorem 8.4.1), convergence in probability is easily found. A Taylor series expansion of $\sum_{t=1}^{N} \mathbf{Z}_t$ around $\boldsymbol{\beta}$ and evaluating at $\hat{\boldsymbol{\beta}}$ gives

$$\sum_{t=1}^{N} \hat{\mathbf{Z}}_t = \sum_{t=1}^{N} \mathbf{Z}_t + (\partial \sum_{t=1}^{N} \mathbf{Z}_t / \partial \boldsymbol{\beta}^\top)(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) + \mathbf{O}_p(1),$$

where $\hat{\mathbf{Z}}_t$ represents the evaluation of $\mathbf{Z}_t$ at $\hat{\boldsymbol{\beta}}$. But $\sum_{t=1}^{N} \hat{\mathbf{Z}}_t = \mathbf{0}$, and it follows that $\mathbf{0} = N^{-1/2} \sum_{t=1}^{N} \mathbf{Z}_t + N^{-1}(\partial \sum_{t=1}^{N} \mathbf{Z}_t / \partial \boldsymbol{\beta}^\top) N^{1/2}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) + \mathbf{o}_p(1)$. Lemma 2.1 and an application of Slutsky theorem allow us to show Theorem 2.1. □

2.3.2. Linear constraints on the model parameters

We now study least squares estimation when the parameters of the SVARMA model satisfy (2.3). Note that the chain rule for vector differentiation gives

$$\partial \boldsymbol{\epsilon}_t / \partial \boldsymbol{\gamma}^\top = (\partial \boldsymbol{\epsilon}_t / \partial \boldsymbol{\beta}^\top)(\partial \boldsymbol{\beta} / \partial \boldsymbol{\gamma}^\top) = (\partial \boldsymbol{\epsilon}_t / \partial \boldsymbol{\beta}^\top)\mathbf{R}$$

(see, e.g., Lütkepohl (2005, Proposition A.1)). Thus the least squares estimators of $\boldsymbol{\gamma}$, denoted $\hat{\boldsymbol{\gamma}}$, are obtained by solving $\partial S_{LS} / \partial \boldsymbol{\gamma}^\top = \mathbf{0}^\top$, that is they are solution of $\sum_{t=1}^{N} \hat{\mathbf{Z}}_t^\top \mathbf{R} = \mathbf{0}^\top$. Reworking the proof of Theorem 2.1, we have

$$N^{-1/2} \sum_{t=1}^{N} \mathbf{R}^\top \mathbf{Z}_t \xrightarrow{d} \mathcal{N}_K(\mathbf{0}, \boldsymbol{\Omega}_{\boldsymbol{\gamma}})$$

and

$$N^{-1} \partial (\sum_{t=1}^{N} \mathbf{R}^\top \mathbf{Z}_t) / \partial \boldsymbol{\gamma}^\top = N^{-1} \mathbf{R}^\top \{\partial (\sum_{t=1}^{N} \mathbf{Z}_t) / \partial \boldsymbol{\beta}^\top\} \mathbf{R} \xrightarrow{p} \boldsymbol{\Omega}_{\boldsymbol{\gamma}},$$

where $\boldsymbol{\Omega}_{\boldsymbol{\gamma}} = \mathbf{R}^\top \boldsymbol{\Omega}_{\boldsymbol{\beta}} \mathbf{R}$. Consequently, under the conditions of Theorem 2.1, the estimator $\hat{\boldsymbol{\gamma}}$ is consistent for $\boldsymbol{\gamma}$ and we have the following corollary.

Corollary 2.1. *Under the conditions of Theorem 2.1, the distributions of the least squares estimators $\hat{\boldsymbol{\gamma}}$ of the model parameters $\boldsymbol{\gamma}$, where $\boldsymbol{\gamma}$ satisfies the linear constraint $\boldsymbol{\beta} = \mathbf{R}\boldsymbol{\gamma} + \mathbf{b}$, are asymptotically normal :*

$$N^{1/2}(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}) \xrightarrow{d} \mathcal{N}_K\left(\mathbf{0}, \boldsymbol{\Omega}_{\boldsymbol{\gamma}}^{-1}\right), \tag{2.10}$$

where $\boldsymbol{\Omega}_{\boldsymbol{\gamma}} = \mathbf{R}^\top \boldsymbol{\Omega}_{\boldsymbol{\beta}} \mathbf{R}$ and $\boldsymbol{\Omega}_{\boldsymbol{\beta}}$ is given by (2.8).

Based on the previous corollary, an estimator of β is given by $\hat{\boldsymbol{\beta}} = \mathbf{R}\hat{\boldsymbol{\gamma}} + \mathbf{b}$, which is normal asymptotically : $N^{1/2}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \xrightarrow{d} \mathcal{N}_{d^2(p+q+P+Q)}\left(\mathbf{0}, \mathbf{R}\boldsymbol{\Omega}_{\boldsymbol{\gamma}}^{-1}\mathbf{R}^\top\right)$. See Reinsel (1997, p. 131) for similar results established in the context of VARMA models. Here, Corollary 2.1 generalizes that kind of results for SVARMA models with linear constraints on the parameters.

2.4. Asymptotic distributions of the residual autocovariance matrices

Let $\hat{\boldsymbol{\epsilon}}_t$, $t = 1, 2, \ldots, N$, be the least squares residuals. We derive in this section the asymptotic distribution of the residual autocovariance matrices, noted $\mathbf{c}_{\hat{\boldsymbol{\epsilon}}}$, in the class of SVARMA models. We consider the full unconstrained case and also the situation where the model parameters satisfy the linear constraints (2.3). Based on these measures of dependence, we study portmanteau test statistics.

2.4.1. Asymptotic distributions of the residual autocovariance in the full unconstrained case

It is well-known that the asymptotic distribution of $\mathbf{c}_{\boldsymbol{\epsilon}}$ follows asymptotically a d^2M-variate normal distribution : $N^{1/2}\mathbf{c}_{\boldsymbol{\epsilon}} \xrightarrow{d} \mathcal{N}_{d^2M}(\mathbf{0}, \mathbf{I}_M \otimes \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}} \otimes \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}})$ (see, e.g., Li and McLeod (1981), Reinsel (1997, p. 151)). Due to the nonseasonal and seasonal polynomials of the autoregressive moving average components, it is useful to write $\boldsymbol{\Omega}_{\boldsymbol{\beta}}$ defined by (2.8) as a block-diagonal matrix $\boldsymbol{\Omega}_{\boldsymbol{\beta}} = (\boldsymbol{\Omega}_{\boldsymbol{\beta},ij})_{i,j=1,\ldots,4}$, where the blocks $\boldsymbol{\Omega}_{\boldsymbol{\beta},ij}$, $i, j \in \{1, 2, 3, 4\}$, are composed of the quantities $\boldsymbol{\Omega}_{\boldsymbol{\beta},ij} = E\{(\partial\boldsymbol{\epsilon}_t^\top/\partial\boldsymbol{\beta}_i)\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\partial\boldsymbol{\epsilon}_t/\partial\boldsymbol{\beta}_j^\top)\}$. For example, some calculations

using relations (2.2) and (2.7) give :

$$\mathbf{\Omega}_{\boldsymbol{\beta},11} = \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \begin{pmatrix} \boldsymbol{\Psi}_k\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}\boldsymbol{\Psi}_{k-(j-i)}^{\top} & \cdots & \boldsymbol{\Psi}_k\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}\boldsymbol{\Psi}_{k-(j-p-i+1)}^{\top} \\ \vdots & \ddots & \vdots \\ \boldsymbol{\Psi}_k\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}\boldsymbol{\Psi}_{k-(j-i-1+p)}^{\top} & \cdots & \boldsymbol{\Psi}_k\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}\boldsymbol{\Psi}_{k-(j-i)}^{\top} \end{pmatrix} \otimes \mathbf{G}_{1i}^{\top}\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}\mathbf{G}_{1j}.$$

Using the asymptotic representation of the least squares estimators (see the proof of Theorem 2.1), it follows that

$$\lim_{N\to\infty} N\mathrm{cov}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}, \mathbf{c}_{\boldsymbol{\epsilon}}) = -\mathbf{\Omega}_{\boldsymbol{\beta}}^{-1} \lim_{N\to\infty} E\left(\sum_{t=1}^{N} \mathbf{Z}_t\mathbf{c}_{\boldsymbol{\epsilon}}^{\top}\right).$$

Given the definition of $\mathbf{Z}_t$, and using the fact that $\mathbf{c}_{\boldsymbol{\epsilon}}^{\top}(l) = N^{-1}\sum_{t=l+1}^{N} \boldsymbol{\epsilon}_{t-l}^{\top} \otimes \boldsymbol{\epsilon}_t^{\top}$, we obtain :

$$\begin{aligned} \lim_{N\to\infty} E\left\{\sum_{t=1}^{N} \mathbf{Z}_t\mathbf{c}_{\boldsymbol{\epsilon}}^{\top}(l)\right\} &= \lim_{N\to\infty} N^{-1}\sum_{t=1}^{N}\sum_{s=l+1}^{N} E\left\{\left(\frac{\partial\boldsymbol{\epsilon}_t^{\top}}{\partial\boldsymbol{\beta}}\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}\boldsymbol{\epsilon}_t\right)\left(\boldsymbol{\epsilon}_{s-l}^{\top} \otimes \boldsymbol{\epsilon}_s^{\top}\right)\right\}, \\ &\equiv \mathbf{H}(l) = (\mathbf{H}_1^{\top}(l), \mathbf{H}_2^{\top}(l), \mathbf{H}_3^{\top}(l), \mathbf{H}_4^{\top}(l))^{\top}, \end{aligned} \tag{2.11}$$

where $\mathbf{H}_1(l)$, $\mathbf{H}_2(l)$, $\mathbf{H}_3(l)$ and $\mathbf{H}_4(l)$ are matrices of dimension $(d^2p)\times d^2$, $(d^2q)\times d^2$, $(d^2P)\times d^2$ and $(d^2Q)\times d^2$, respectively. For example, using again the infinite moving average representation (2.2) gives :

$$\begin{aligned} \mathbf{H}_1(l) &\equiv -\lim_{N\to\infty} N^{-1}\sum_{t=1}^{N}\sum_{s=l+1}^{N} E\left[\left(\sum_{i=0}^{\infty}\boldsymbol{\zeta}_p(B)\otimes\mathbf{Y}_t\otimes\mathbf{G}_{1i}^{\top}\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}\boldsymbol{\epsilon}_t\right)\left\{\boldsymbol{\epsilon}_{s-l}^{\top}\otimes\boldsymbol{\epsilon}_s^{\top}\right\}\right], \\ &= -\sum_{i=0}^{\infty}\left\{\boldsymbol{\zeta}_p(B)\otimes\boldsymbol{\Psi}_{l-i}\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}\otimes\mathbf{G}_{1i}^{\top}\right\}, \end{aligned} \tag{2.12}$$

where in the last equality the backward shift operator acts on the moving average weights $\boldsymbol{\Psi}_j$, that is $B\boldsymbol{\Psi}_j = \boldsymbol{\Psi}_{j-1}$. These results lead us to Theorem 2.2.

Theorem 2.2. *The asymptotic distributions of the residual autocovariance matrices are given by :*

$$N^{1/2}\mathbf{c}_{\hat{\boldsymbol{\epsilon}}} \xrightarrow{d} \mathcal{N}_{d^2M}\left(\mathbf{0}, \mathbf{\Upsilon}_{\boldsymbol{\beta}}\right), \tag{2.13}$$

where $\mathbf{\Upsilon}_{\boldsymbol{\beta}} = \mathbf{I}_M \otimes \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}} \otimes \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}} - \mathbf{H}^{\top}\mathbf{\Omega}_{\boldsymbol{\beta}}^{-1}\mathbf{H}$. *The* $\{d^2(p+q+P+Q)\}\times(d^2M)$ *matrix* $\mathbf{H}$ *satisfies* $\mathbf{H} = (\mathbf{H}(1),\ldots,\mathbf{H}(M))$, *where* $\mathbf{H}(l)$, $l = 1,\ldots,M$ *are defined by (2.11).*

PROOF. To show the theorem, a Taylor series expansion of $\mathbf{c}_{\hat{\boldsymbol{\epsilon}}}$ around $\boldsymbol{\beta}$ is performed, as in Ursu and Duchesne (2008). A careful determination of the gradient in the first order term allows us to show that $\partial \mathbf{c}_{\hat{\boldsymbol{\epsilon}}}/\partial \boldsymbol{\beta}^\top \xrightarrow{p} -\mathbf{H}^\top$. The theorem easily follows, noting that $N^{1/2}\mathbf{c}_{\hat{\boldsymbol{\epsilon}}}$ and $N^{1/2}\{\mathbf{c}_{\boldsymbol{\epsilon}} - \mathbf{H}^\top(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})\}$ have the same asymptotic distribution. □

Theorem 2.2 generalizes for vector time series a result of McLeod (1978) for SARMA models. See also Li (2004, p. 13). The theorem gives also new asymptotic results of the residual autocovariance matrices for vector time series. For VARMA models, see Hosking (1980), Li and McLeod (1981), Poskitt and Tremayne (1982), Li (2004, Chapter 3) and Lütkepohl (2005, Chapter 13).

2.4.2. Asymptotic distributions of the residual autocovariances when the parameters satisfy linear constraints

When the parameters satisfy linear constraints, an application of the chain rule for vector differentiation gives

$$\lim_{N\to\infty} E\left[\sum_{t=1}^{N}\{(\partial \boldsymbol{\epsilon}_t^\top/\partial \boldsymbol{\gamma})\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}\boldsymbol{\epsilon}_t\}\mathbf{c}_{\boldsymbol{\epsilon}}^\top(l)\right] = \mathbf{R}^\top\mathbf{H}(l).$$

Furthermore, proceeding as in the proof of Theorem 2.2, we obtain

$$\partial \mathbf{c}_{\hat{\boldsymbol{\epsilon}}}/\partial \boldsymbol{\gamma}^\top \xrightarrow{p} -\mathbf{H}^\top\mathbf{R}.$$

Thus, the asymptotic distribution of the residual autocovariances when the parameters satisfy the linear constraints (2.3) is easily deduced. The result is stated more precisely in Corollary 2.2.

Corollary 2.2. *The asymptotic distributions of the residual autocovariance matrices when the model parameters $\boldsymbol{\gamma}$ satisfy the linear constraint $\boldsymbol{\beta} = \mathbf{R}\boldsymbol{\gamma} + \mathbf{b}$ are normal asymptotically. More precisely,*

$$N^{1/2}\mathbf{c}_{\hat{\boldsymbol{\epsilon}}} \xrightarrow{d} \mathcal{N}_{d^2M}\left(\mathbf{0}, \boldsymbol{\Upsilon}_{\boldsymbol{\gamma}}\right), \tag{2.14}$$

where $\boldsymbol{\Upsilon}_{\boldsymbol{\gamma}} = \mathbf{I}_M \otimes \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}} \otimes \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}} - \mathbf{H}^\top\mathbf{R}\boldsymbol{\Omega}_{\boldsymbol{\gamma}}^{-1}\mathbf{R}^\top\mathbf{H}$.

Corollary 2.2 generalizes the theorems of Lütkepohl (2005) in VAR models when the model parameters satisfy linear constraints, and those of Reinsel (1997)

in VARMA models. Note that Theorem 2.2 and Corollary 2.2 concentrate on the asymptotic distributions of the residual autocovariance matrices. The asymptotic distributions of the residual autocorrelation matrices are obtained by an appropriate scaling of results (2.13) and (2.14) (see, e.g., Ursu and Duchesne (2008)).

2.4.3. Applications : portmanteau test statistics

We consider the following portmanteau test statistics :

$$\mathcal{Q}_M = N\sum_{l=1}^{M}\mathrm{tr}\{\mathbf{C}_{\hat{\boldsymbol{\epsilon}}}^{\top}(l)\hat{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon}}^{-1}\mathbf{C}_{\hat{\boldsymbol{\epsilon}}}(l)\hat{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon}}^{-1}\}, \tag{2.15}$$

$$\mathcal{Q}_M^* = N\sum_{l=1}^{M}\frac{N}{N-l}\mathrm{tr}\{\mathbf{C}_{\hat{\boldsymbol{\epsilon}}}^{\top}(l)\hat{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon}}^{-1}\mathbf{C}_{\hat{\boldsymbol{\epsilon}}}(l)\hat{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon}}^{-1}\}, \tag{2.16}$$

where M is the maximal lag order and $\hat{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon}} = \mathbf{C}_{\hat{\boldsymbol{\epsilon}}}(0)$.

The test statistics Q_M and Q_M^* represent the multivariate generalization of the Box-Pierce and Ljung-Box test statistics, respectively. The correction factor $N/(N-l)$ in Q_M^* generally improves the finite samples properties. See Li (2004), for an introduction to Box-Pierce and Ljung-Box test statistics. Portmanteau test statistics have been studied for VARMA models by many authors (see, e.g., Hosking (1980), Li and McLeod (1981) and Poskitt and Tremayne (1982), Li (2004) and Lütkepohl (2005)).

In order to study the distributions of (2.15) and (2.16) for SVARMA models, we introduce the $d \times d$ matrix $\mathbf{P}$, satisfying $\mathbf{P}\mathbf{P}^{\top} = \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}$, $\mathbf{P}\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}\mathbf{P}^{\top} = \mathbf{I}_d$, and the matrix $\mathbf{Q}_M$, defined as $\mathbf{Q}_M = \mathbf{I}_M \otimes \mathbf{P} \otimes \mathbf{P}$. Using the transformation $\tilde{\mathbf{c}}_{\hat{\boldsymbol{\epsilon}}} = \mathbf{Q}_M \mathbf{c}_{\hat{\boldsymbol{\epsilon}}}$, it follows that the asymptotic covariance matrix of $\tilde{\mathbf{c}}_{\hat{\boldsymbol{\epsilon}}}$ is

$$\lim_{N\to\infty} N\mathrm{var}\{\tilde{\mathbf{c}}_{\hat{\boldsymbol{\epsilon}}}\} = \mathbf{I}_{d^2M} - \mathbf{Q}_M\left\{\mathbf{H}^{\top}\boldsymbol{\Omega}^{-1}\mathbf{H}\right\}\mathbf{Q}_M^{\top}.$$

Proceeding as in Li and McLeod (1981), the following relation holds approximately :

$$\boldsymbol{\Omega}_{\boldsymbol{\beta}} \approx \mathbf{H}(\mathbf{I}_M \otimes \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1} \otimes \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1})\mathbf{H}^{\top},$$

where M is chosen large enough so that $||\boldsymbol{\Psi}_k|| \cong 0$ when $k > M$. Consequently, the asymptotic covariance matrix $N\mathrm{var}\{\tilde{\mathbf{c}}_{\hat{\boldsymbol{\epsilon}}}\}$ is approximately idempotent of rank

$d^2\{M-(p+q+P+Q)\}$. To illustrate the result, consider the block $\boldsymbol{\Omega}_{\boldsymbol{\beta},11}$ and the corresponding block matrix in $\mathbf{H}\{\mathbf{I}_M \otimes \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1} \otimes \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}\}\mathbf{H}^\top$. Using (2.12), simple algebra yields :

$$\sum_{l=1}^{M} \mathbf{H}_1(l)(\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1} \otimes \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1})\mathbf{H}_1^\top(l) =$$

$$\sum_{l=1}^{M}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \begin{pmatrix} \boldsymbol{\Psi}_{l-i-1}\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}\boldsymbol{\Psi}_{l-j-1}^\top & \cdots & \boldsymbol{\Psi}_{l-i-1}\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}\boldsymbol{\Psi}_{l-j-p}^\top \\ \vdots & \ddots & \vdots \\ \boldsymbol{\Psi}_{l-i-p}\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}\boldsymbol{\Psi}_{l-j-1}^\top & \cdots & \boldsymbol{\Psi}_{l-i-p}\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}\boldsymbol{\Psi}_{l-j-p}^\top \end{pmatrix} \otimes \mathbf{G}_{1i}^\top \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}\mathbf{G}_{1j}.$$

From the preceding derivation we can conclude the approximate equality with the matrix $\boldsymbol{\Omega}_{\boldsymbol{\beta},11}$, which is improved when M is chosen large enough. The other blocks are done similarly. The mathematical details are given in a technical report available upon request by contacting the authors.

Under the linear constraints (2.3), the arguments are similar. Premultiplying $\boldsymbol{\Omega}_{\boldsymbol{\beta}}$ by $\mathbf{R}^\top$, and postmultiplying by $\mathbf{R}$, it follows that

$$\boldsymbol{\Omega}_{\boldsymbol{\gamma}} = \mathbf{R}^\top \boldsymbol{\Omega}_{\boldsymbol{\beta}} \mathbf{R} \approx \mathbf{R}^\top \mathbf{H}\{\mathbf{I}_M \otimes \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1} \otimes \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}\}\mathbf{H}^\top \mathbf{R}$$

holds approximately. Using the asymptotic result (2.14) and the transformation $\tilde{\mathbf{c}}_{\hat{\boldsymbol{\epsilon}}} = \mathbf{Q}_M \mathbf{c}_{\hat{\boldsymbol{\epsilon}}}$, we deduce that the asymptotic covariance matrix

$$\lim_{N\to\infty} N\mathrm{var}(\tilde{\mathbf{c}}_{\hat{\boldsymbol{\epsilon}}}) = \mathbf{I}_{d^2M} - \mathbf{Q}_M \left\{\mathbf{H}^\top \mathbf{R} \boldsymbol{\Omega}_{\boldsymbol{\gamma}}^{-1} \mathbf{R}^\top \mathbf{H}\right\} \mathbf{Q}_M^\top$$

is approximatively idempotent of rank $(d^2M) - K$.

Under the null hypothesis of adequacy of a particular SVARMA$(p,q)\times(P,Q)_s$ model, $N\tilde{\mathbf{c}}_{\hat{\boldsymbol{\epsilon}}}^\top \tilde{\mathbf{c}}_{\hat{\boldsymbol{\epsilon}}} = Q_M + o_P(1)$, and $Q_M = Q_M^* + o_P(1)$. Thus the test statistics $\mathcal{Q}_M$ and $\mathcal{Q}_M^*$ follow approximatively a chi-square distribution $\chi^2_{d^2\{M-(p+q+P+Q)\}}$. When the model parameters are supposed to satisfy the linear constraint (2.3), the asymptotic distributions of the test statistics (2.15) and (2.16) are approximatively chi-square $\chi^2_{d^2M-K}$. These test statistics represent natural multivariate versions of test statistics originally proposed by McLeod (1978) in univariate SARMA models. See also Li (2004, p. 13). For VARMA models with linear constraints on the parameters, see Lütkepohl (2005, p. 510). In the next section, a small Monte Carlo study is conducted in order to study the finite sample properties of Q_M

and Q_M^*.

2.5. Simulation experiments

In the previous section, we presented portmanteau test statistisc for checking the overall significance of the residual autocovariances of a SVARMA$(p,q) \times (P,Q)_s$ model. To appreciate the finite sample properties of these test procedures, we report the simulation results of a small Monte Carlo experiment conducted in order to study their exact levels. To compare the exact distribution of the test statistics with their corresponding χ^2 distributions, six bivariate data generating processes (DGP) were used :

$$\begin{aligned}
\text{DGP}_1 &: (\mathbf{I} - \mathbf{\Phi}_1 B)\mathbf{Y}_t = (\mathbf{I} - \mathbf{\Xi}_1 B^4)\boldsymbol{\epsilon}_t, \\
\text{DGP}_2 &: (\mathbf{I} - \mathbf{\Phi}_{2C} B)\mathbf{Y}_t = (\mathbf{I} - \mathbf{\Xi}_{2C} B^4)\boldsymbol{\epsilon}_t, \\
\text{DGP}_3 &: (\mathbf{I} - \mathbf{\Lambda}_{3C} B^4)\mathbf{Y}_t = (\mathbf{I} - \mathbf{\Theta}_{3C} B)\boldsymbol{\epsilon}_t, \\
\text{DGP}_4 &: (\mathbf{I} - \mathbf{\Lambda}_{4C} B^{12})\mathbf{Y}_t = (\mathbf{I} - \mathbf{\Theta}_{4C} B)\boldsymbol{\epsilon}_t, \\
\text{DGP}_5 &: (\mathbf{I} - \mathbf{\Lambda}_{5C} B^4)(\mathbf{I} - \mathbf{\Phi}_{5C} B)\mathbf{Y}_t = \boldsymbol{\epsilon}_t, \\
\text{DGP}_6 &: (\mathbf{I} - \mathbf{\Lambda}_6 B^4)(\mathbf{I} - \mathbf{\Phi}_6 B)\mathbf{Y}_t = (\mathbf{I} - \mathbf{\Xi}_6 B^4)\boldsymbol{\epsilon}_t.
\end{aligned}$$

The models DGP_1 and DGP_2 can be written as SVARMA$(1,0) \times (0,1)_4$ models. The case DGP_3 corresponds to a SVARMA$(0,1) \times (1,0)_4$, which may be suitable for quarterly data. The monthly version, that is a SVARMA$(0,1) \times (1,0)_{12}$ model, is the case DGP_4. Under DGP_5, the multiplicative seasonal autoregressive model SVARMA$(1,0) \times (1,0)_4$ is generated. Finally, DGP_6 represents a multiplicative seasonal model SVARMA$(1,0) \times (1,1)_4$. The model coefficients of DGP_i, $i \in \{1,2,3,4,5,6\}$, are given in Table 1. For each DGP, the stochastic process $\boldsymbol{\epsilon} = \{\boldsymbol{\epsilon}_t, t \in \mathbb{Z}\}$ was assumed to be a Gaussian white noise, composed of independent Gaussian random vectors with mean $\mathbf{0}$ and covariance matrix $\mathbf{\Sigma}_{\boldsymbol{\epsilon}}$, where the covariance matrix satisfies :

$$\mathbf{\Sigma}_{\boldsymbol{\epsilon}} = \begin{pmatrix} 0.90 & 0.60 \\ 0.60 & 1.70 \end{pmatrix}.$$

For the purposes of our illustration, no parameter constraints were hypothesized for DGP_1 and DGP_6, but for DGP_i, $i \in \{2,3,4,5\}$, it was assumed that the zero-valued parameters were known.

We examined the empirical frequencies of rejection of the null hypothesis of adequacy at two different nominal levels (5 and 10 percent) for time series of moderate length ($N = 400$ observations). A total of 1000 independent realizations were generated. For each realization of DGP_1 and DGP_6, a SVARMA model was estimated by least squares estimators, as described in Section 2.3.1. Under DGP_2, DGP_3, DGP_4 and DGP_5, the zero-valued parameters were taken into account by properly defining the constraint matrix $\mathbf{R}$ and setting $\mathbf{b} = \mathbf{0}$. Then, the parameters were estimated using the procedure described in Section 2.3.2. For each residual time series, the portmanteau test statistics Q_M and Q_M^* were calculated for $M = 15, 20, 25, 30, 35, 40$. For each nominal level, we obtained from the 1000 realizations the empirical frequencies of rejection of the null hypothesis of adequacy. The standard errors of the empirical levels based on 1000 independent

TABLE 2.1. Model parameters for DGP_i, $i \in \{1,2,3,4,5,6\}$.

$$\boldsymbol{\Phi}_1 = \begin{pmatrix} 0.90 & 0.20 \\ 0.30 & -0.50 \end{pmatrix}, \qquad \boldsymbol{\Xi}_1 = \begin{pmatrix} 0.30 & -0.40 \\ -0.20 & -0.25 \end{pmatrix},$$

$$\boldsymbol{\Phi}_{2C} = \begin{pmatrix} 0.90 & 0 \\ 0.30 & -0.50 \end{pmatrix}, \qquad \boldsymbol{\Xi}_{2C} = \begin{pmatrix} 0.30 & -0.40 \\ 0 & -0.25 \end{pmatrix},$$

$$\boldsymbol{\Lambda}_{3C} = \boldsymbol{\Lambda}_{4C} = \begin{pmatrix} 0.40 & 0 \\ -0.30 & 0.70 \end{pmatrix}, \qquad \boldsymbol{\Theta}_{3C} = \boldsymbol{\Theta}_{4C} = \begin{pmatrix} -0.50 & -0.80 \\ 0 & -0.60 \end{pmatrix},$$

$$\boldsymbol{\Lambda}_{5C} = \begin{pmatrix} 0 & 0.20 \\ 0 & -0.50 \end{pmatrix}, \qquad \boldsymbol{\Phi}_{5C} = \begin{pmatrix} -0.50 & 0 \\ 0.30 & 0 \end{pmatrix},$$

$$\boldsymbol{\Lambda}_6 = \begin{pmatrix} 0.70 & 0.20 \\ 0.30 & -0.50 \end{pmatrix}, \qquad \boldsymbol{\Phi}_6 = \begin{pmatrix} -0.50 & -0.60 \\ 0.30 & -0.70 \end{pmatrix},$$

$$\boldsymbol{\Xi}_6 = \begin{pmatrix} 0.60 & -0.40 \\ -0.70 & -0.50 \end{pmatrix}.$$

realizations are 0.689% and 0.948% for the nominal levels 5% and 10%, respectively. All the computer code has been written using the MATLAB software. For estimating VARMA models, we implemented the algorithms based on the E^4 package developed by Terceiro (1990) and Casals, Sotoca and Jerez (1999), and we also used the efficient estimation procedures described in Jonasson (2008) and Jonasson and Ferrando (2008).

The empirical levels of $\mathcal{Q}_M$ and $\mathcal{Q}_M^*$ for the SVARMA models with and without parameter constraints are presented in Table 2. As expected, the test statistics $\mathcal{Q}_M^*$ exhibited better empirical levels than $\mathcal{Q}_M$. As for VARMA models, the Ljung-Box factor correction improved the χ^2 approximation for the test statistic $\mathcal{Q}_M^*$, offering generally better finite sample properties than $\mathcal{Q}_M$, particularly as M increases. We concentrate the rest of our discussion on $\mathcal{Q}_M^*$ only. Generally, the χ^2 distribution

TABLE 2.2. Empirical levels (in percentage) of the portmanteau test statistics Q_M and Q_M^*, defined by (2.15) and (2.16), respectively, for the DGP$_i$, $i \in \{1,2,3,4,5,6\}$.

	$\alpha = 0.05$		$\alpha = 0.10$		$\alpha = 0.05$		$\alpha = 0.10$		$\alpha = 0.05$		$\alpha = 0.10$	
M	Q_M	Q_M^*	Q_M	Q_M^*	Q_M	Q_M^*	Q_M	Q_M^*	Q_M	Q_M^*	Q_M	Q_M^*
	DGP$_1$				DGP$_2$				DGP$_3$			
15	4.5	5.6	9.7	11.4	4.1	5.0	7.2	9.5	3.9	4.8	7.3	9.3
20	3.7	5.0	7.9	10.2	3.2	4.6	7.6	9.4	3.5	4.9	7.7	10.5
25	3.6	5.4	6.5	9.7	3.0	4.7	6.1	8.9	2.5	4.7	6.8	9.4
30	3.2	4.8	5.8	9.6	2.7	4.0	4.9	9.2	2.4	5.7	6.3	9.2
35	2.4	5.8	5.9	10.8	2.4	5.3	5.3	10.5	2.2	4.7	4.8	9.0
40	2.0	5.5	4.2	9.1	2.0	4.3	4.0	9.3	1.7	5.2	4.8	9.1
	DGP$_4$				DGP$_5$				DGP$_6$			
15	3.9	4.5	8.6	10.8	4.6	5.9	8.5	11.2	3.5	5.5	8.3	10.0
20	3.3	4.5	7.7	9.7	3.2	5.2	7.2	11.2	3.8	5.5	6.6	9.9
25	3.6	5.3	6.4	9.0	2.7	5.9	6.8	11.4	3.1	5.6	6.3	10.0
30	3.0	5.2	6.3	9.0	2.4	5.7	5.7	11.1	2.6	6.1	5.9	11.1
35	1.6	4.6	4.6	9.8	2.0	5.1	4.6	11.4	2.6	6.0	5.5	12.2
40	1.7	4.4	4.1	8.9	2.5	6.6	5.0	12.2	3.0	8.0	5.6	14.0

provided a satisfactory approximation for all lags, at both significance levels. The results for the models without and with parameter constraints were very comparable. In general, the rejection rates of the test statistic Q_M^* at the 5% and 10% nominal levels lie within the 5% significant limits, or they are reasonably close to these intervals. It should be noted that some overrejection has been observed under multiplicative models, particularly for large values of M. This suggests that large sample sizes may be needed for complex time series models with a multiplicative structure.

From this limited empirical study, the finite sample performance of Q_M^* seem rather reasonable, and it can be recommended for diagnosing SVARMA models. Overall, it is hoped that the results presented in this paper will be useful in practice, complementing periodic models, especially in finding and diagnosing parsimonious representations of seasonal vector time series models.

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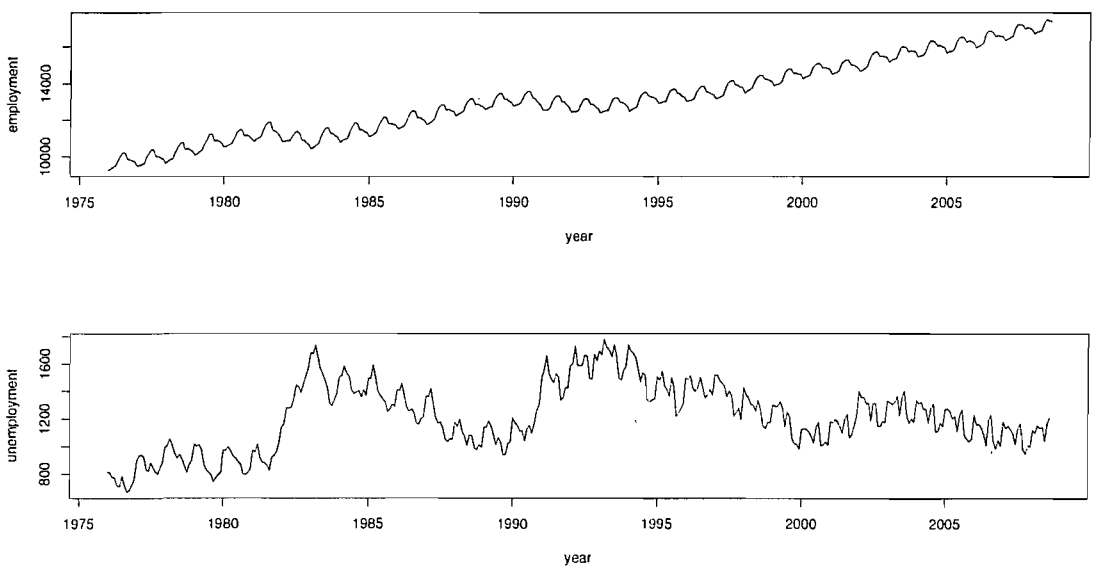
2.7. APPENDIX

APPLICATION USING A BIVARIATE QUARTERLY DATA SET ON BASIC LABOUR FORCE FOR CANADA (Persons aged 15+, both sexes)

In this section, the proposed methodology is illustrated with two univariate time series data coming from the Canadian Socio-Economic Information Management System of Statistics Canada. The bivariate data set is composed of the total number of employed and unemployed person for 15 years and over, both sexes in Canada. The data are monthly, not seasonally adjusted and the period ranges from January 1976 to August 2008.

The original data are represented in Figure 2.1. The Box-Cox transformation has been considered for each variable. The 0.5 value was included in the 95% confidence interval for the transformation parameter, suggesting a square root transformation.

FIGURE 2.1. Total number (thousands) of employed (top) and unemployed (bottom) persons, from January 1976 to August 2008.



A double filter, that is a seasonal $(\mathbf{I}_d - B^{12})$ and a first order differencing filter $(\mathbf{I}_d - B)$, is often applied to model nonstationary seasonal time series. In practice, only the seasonal filter is expected to be useful and theoretical arguments suggest to avoid double differencing (see Osborn (1990) and Franses (1996, p. 63)). The same results were obtained by Beaulieu and Miron (1993).

Two tests often used to verify the existence of unit roots are the HEGY test (see Hylleberg, Engle, Granger and Yoo (1990)) and the CH test (Canova and Hansen (1995)). We used the CH test because the rejection of the null hypothesis implies that the series has a seasonal pattern. Also, if the process contains a moving average component (see below), the HEGY test may be seriously affected by the autocorrelations in the errors (see Reinsel (1997, p. 225) and Hylleberg (2006)). The tests for instability at each seasonal frequencies and the joint test at all seasonal frequency appear in Table 2.3. These results indicate the existence

FIGURE 2.2. Seasonal differences for the number of employed (top) and unemployed (bottom) persons (in thousands), from January 1976 to August 2008.

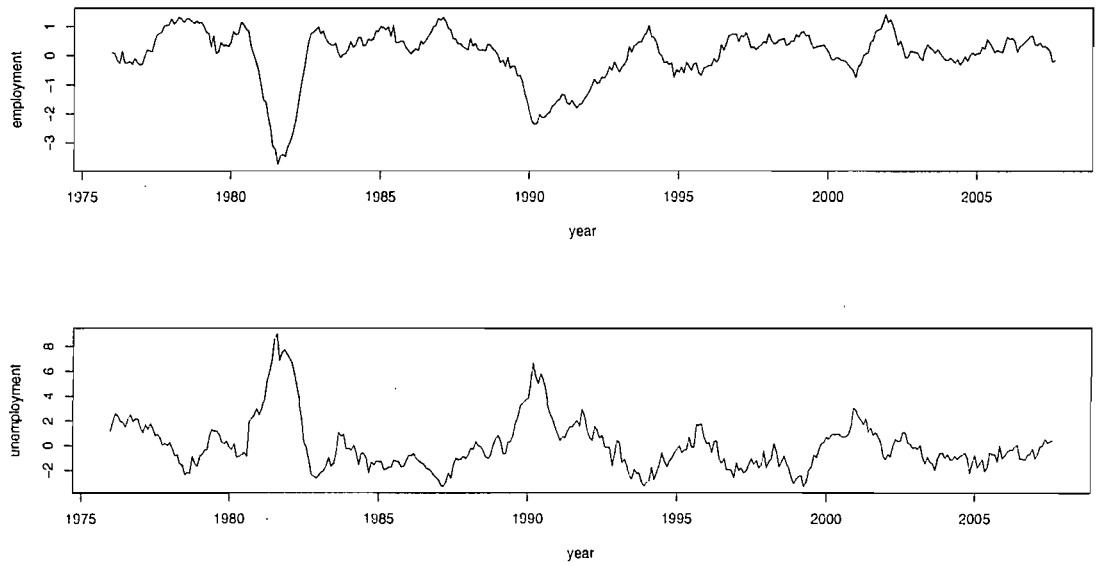


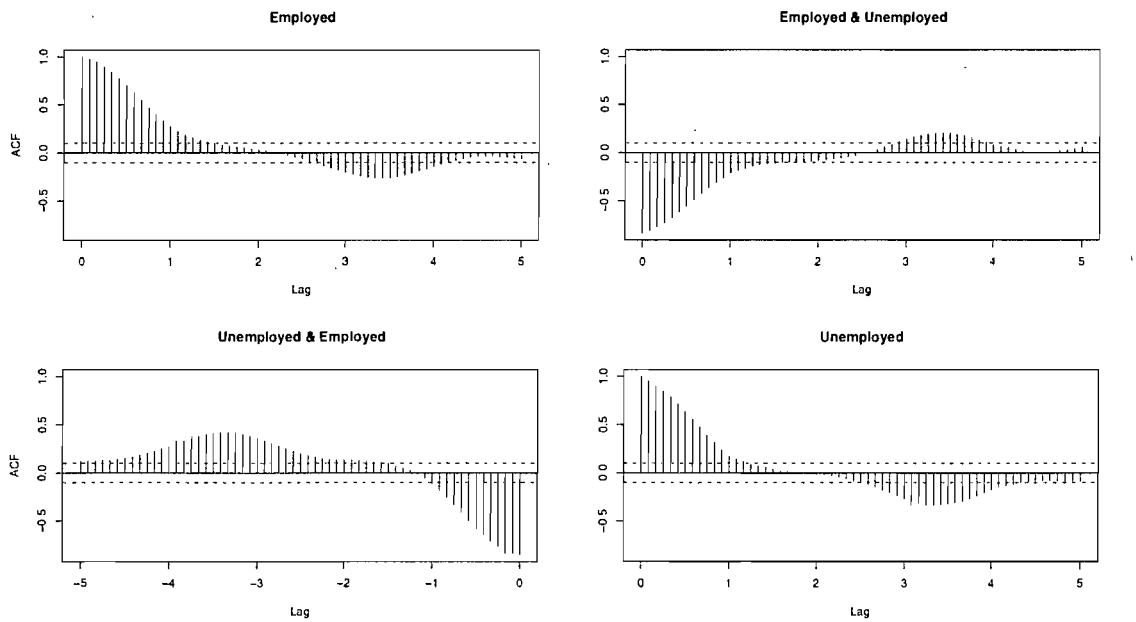
TABLE 2.3. The results for CH test for seasonal unit roots in monthly series (labour force for Canada). An asterisk indicates significance at 5% level.

Series	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π	joint
employed	1.997*	1.811*	2.025*	1.511*	1.218*	0.034	2.982*
unemployed	1.490*	1.117*	1.260*	1.919*	1.045*	0.481*	2.875*

of seasonal unit roots, and consequently the seasonal difference $(\mathbf{I}_d - B^{12})$ is considered for modeling. The seasonal differences are shown in Figure 2.2.

The sample ACF in Figure 2.3 displays a clear sine-cosine phenomenon indicating a VAR model of high order (see Wei (2006, p. 109)). The PACF in Figure 2.4 strongly suggests $p = 4$. Examinations of the autocorrelation matrices from a VAR(4) model fit to differenced data (Figure 2.5) reveal significant correlations at the seasonal lag 12. In the bivariate model, this suggests the inclusion of a moving average term at lag 12 (see Reinsel (1997, p. 224) or Wei (2006, p. 179)).

FIGURE 2.3. Sample ACF for employed and unemployed persons in Canada, from January 1976 to August 2008.

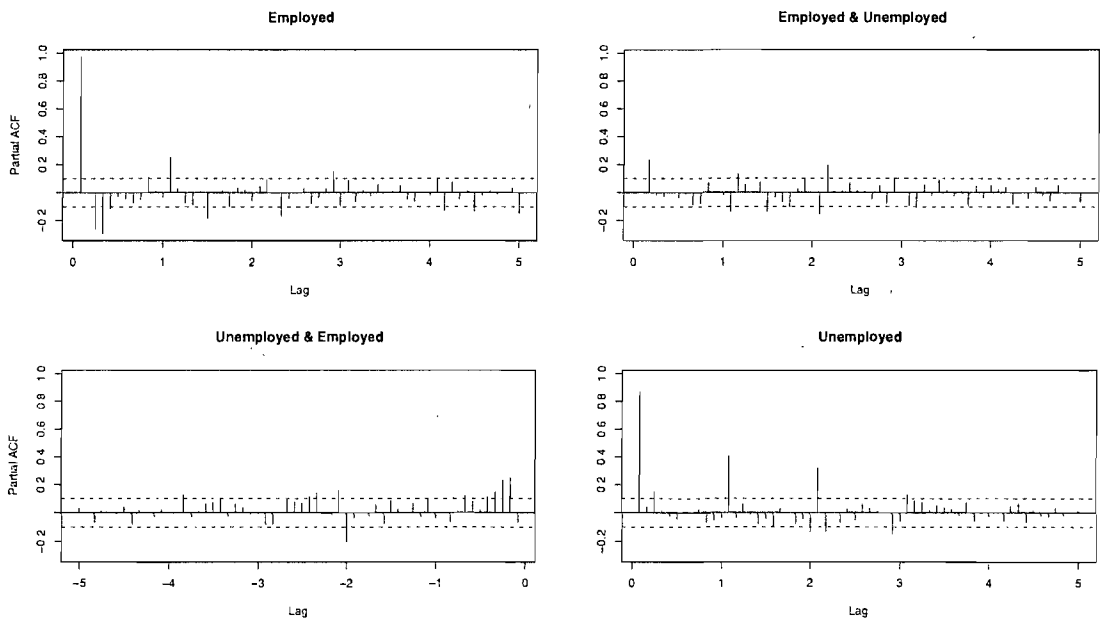


Hence, the model of the form

$$(\mathbf{I} - \mathbf{\Phi}_1 B - \mathbf{\Phi}_2 B^2 - \mathbf{\Phi}_3 B^3 - \mathbf{\Phi}_4 B^4)(\mathbf{I} - B^{12})\mathbf{Y}_t = (\mathbf{I} - \boldsymbol{\theta} B^{12})\boldsymbol{\epsilon}_t \qquad (2.17)$$

has been estimated. First, a full unconstrained model has been estimated. A residual analysis has been made and the portmanteau test statistics $\mathcal{Q}_M$ and $\mathcal{Q}_M^*$ were calculated. Generally, the model was satisfactory, relying on 20 parameters. The maximum likelihood estimators are (the standard errors are given in

FIGURE 2.4. Sample PACF for employed and unemployed persons in Canada, from January 1976 to August 2008.



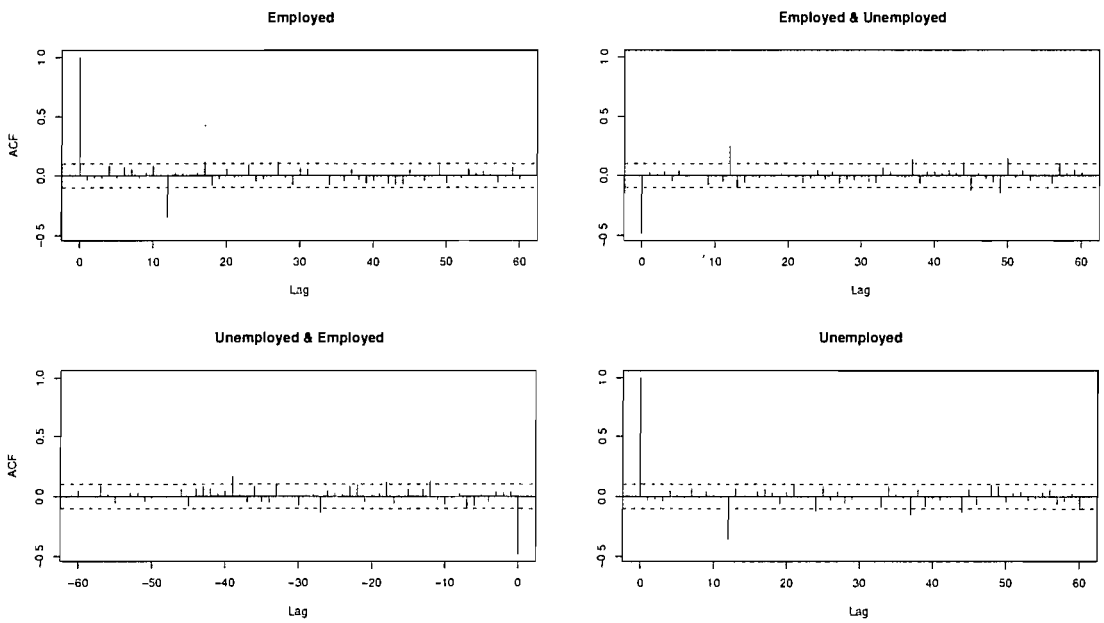
parentheses) :

$$\hat{\mathbf{\Phi}}_1 = \begin{pmatrix} 0.9285 & -0.0498 \\ (0.0736) & (0.0228) \\ -0.7761 & 0.8136 \\ (0.2010) & (0.0601) \end{pmatrix}, \hat{\mathbf{\Phi}}_2 = \begin{pmatrix} 0.3123 & 0.0637 \\ (0.0958) & (0.0263) \\ 0.0457 & -0.0833 \\ (0.2540) & (0.0705) \end{pmatrix},$$

$$\hat{\mathbf{\Phi}}_3 = \begin{pmatrix} 0.0046 & -0.0082 \\ (0.0966) & (0.0259) \\ 0.2508 & 0.2408 \\ (0.2556) & (0.0338) \end{pmatrix}, \hat{\mathbf{\Phi}}_4 = \begin{pmatrix} -0.2670 & -0.0085 \\ (0.0770) & (0.0220) \\ 0.5199 & 0.0097 \\ (0.2078) & (0.0591) \end{pmatrix},$$

$$\hat{\boldsymbol{\theta}} = \begin{pmatrix} -0.5397 & 0.0375 \\ (0.1024) & (0.0324) \\ -0.3428 & -0.7477 \\ (0.2878) & (0.0880) \end{pmatrix}.$$

FIGURE 2.5. Sample ACF of the residual from a VAR(4) model fit to differenced data.



The P-values for diagnosing the estimated model are presented in Table 2.4. In order to propose a more parsimonious model, each parameter whose absolute values of the t-statistic (calculated as the value of the estimator divided by its standard error) was smaller than one was set to zero. This linear constraints can be tested using the likelihood-ratio (LR) procedure described in Reinsel (1997, page 132). The reduced SVARMA model with the following constraints on the parameters $\Phi_{2,21} = 0$, $\Phi_{3,11} = 0$, $\Phi_{3,12} = 0$, $\Phi_{3,21} = 0$, $\Phi_{4,12} = 0$, $\Phi_{4,22} = 0$ and $\theta_{12} = 0$ was estimated. The linear constraint is appropriate if the test statistic :

$$-n \log \frac{\det \hat{\Sigma}}{\det \hat{\Sigma}_C}$$

is asymptotically distributed as chi-squared with 6 degrees of freedom, where $\hat{\Sigma}$ denotes the unrestricted estimate of Σ, $\hat{\Sigma}_C$ denotes the estimate of Σ under the linear constraints and n represents the number of observations. Thus, we find that the LR statistics is 2.6469 with p-value 0.8517. The maximum likelihood

TABLE 2.4. P-values of the portmanteau test statistics defined by (2.15) and (2.16), in adjusting the time series data on the labour force, using a bivariate SVARMA$(4,0) \times (0,1)_{12}$. The seasonal difference filter is applied to each time series.

M	Q_M	Q_M^*
15	0.0678	0.0497
20	0.2053	0.1506
25	0.1390	0.0821
30	0.1241	0.0601
35	0.2961	0.1601
40	0.1061	0.0313

constrained estimators described above are :

$$\hat{\mathbf{\Phi}}_1 = \begin{pmatrix} 0.9061 & -0.0553 \\ (0.0660) & (0.0215) \\ -0.6177 & 0.8387 \\ (0.1174) & (0.0542) \end{pmatrix}, \hat{\mathbf{\Phi}}_2 = \begin{pmatrix} 0.3129 & 0.0531 \\ (0.0737) & (0.0206) \\ 0 & -0.0758 \\ & (0.0565) \end{pmatrix},$$

$$\hat{\mathbf{\Phi}}_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0.2172 \\ & (0.0464) \end{pmatrix}, \hat{\mathbf{\Phi}}_4 = \begin{pmatrix} -0.2474 & 0 \\ (0.0421) & \\ 0.6577 & 0 \\ (0.1103) & \end{pmatrix},$$

$$\hat{\boldsymbol{\theta}} = \begin{pmatrix} -0.5412 & 0.0362 \\ (0.0979) & (0.0319) \\ -0.3771 & -0.7570 \\ (0.2760) & (0.0867) \end{pmatrix}.$$

The residual analysis was done and the portmanteau test statistics $\mathcal{Q}_M$ and $\mathcal{Q}_M^*$ were calculated. The P-values for diagnosing the constrained estimated model are presented in Table 2.5. From Table 2.5, all the P-values suggest that the model was not rejected for the usual significance levels.

TABLE 2.5. P-values of the portmanteau test statistics defined by (2.15) and (2.16), in adjusting the time series data on the labour force, using a bivariate SVARMA$(4,0)\times(0,1)_{12}$ with constraints on the parameters $\Phi_{2,21}=0$, $\Phi_{3,11}=0$, $\Phi_{3,12}=0$, $\Phi_{3,21}=0$, $\Phi_{4,12}=0$, $\Phi_{4,22}=0$ and $\theta_{12}=0$

M	$\mathcal{Q}_M$	$\mathcal{Q}_M^*$
15	0.1567	0.1220
20	0.2878	0.2189
25	0.1877	0.1154
30	0.1818	0.0953
35	0.3965	0.2365
40	0.1610	0.0547

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Chapitre 3

ON MODELING AND DIAGNOSTIC CHECKING OF VECTOR PERIODIC AUTOREGRESSIVE TIME SERIES MODELS

Cet article a été publié dans la revue *Journal of Time Series Analysis* en 2009. Le premier auteur est Eugen Ursu et le coauteur est le directeur de recherche Pierre Duchesne.

Abstract : Vector periodic autoregressive time series models (PVAR) form an important class of time series for modeling data derived from climatology, hydrology, economics, and electrical engineering, amongst others. In this article, we derive the asymptotic distributions of the least squares estimators of the model parameters in PVAR models, allowing the parameters in a given season to satisfy linear constraints. Residual autocorrelations from classical vector autoregressive and moving average models have been found useful for checking the adequacy of a particular model. In view of this, we obtain the asymptotic distribution of the residual autocovariance matrices in the class of PVAR models, and the asymptotic distribution of the residual autocorrelation matrices is given as a corollary. Portmanteau test statistics designed for diagnosing the adequacy of PVAR models are introduced and we study their asymptotic distributions. The proposed test statistics are illustrated in a small simulation study, and an application with bivariate quarterly West German data is presented.

Key words and phrases : Diagnostic checking ; parameter constraints ; periodic time series ; portmanteau test statistics ; residual autocorrelation and autocovariance matrices ; vector time series.

Mathematics subject classification codes (2000) : primary 62M10 ; secondary 62H10.

3.1. INTRODUCTION

Let $\mathbf{Y} = \{\mathbf{Y}_t, t \in \mathbb{Z}\}$ be a vector periodic autoregressive (PVAR) stochastic process :

$$\mathbf{Y}_{ns+\nu} = \sum_{k=1}^{p(\nu)} \mathbf{\Phi}_k(\nu)\mathbf{Y}_{ns+\nu-k} + \boldsymbol{\epsilon}_{ns+\nu}, \tag{3.1}$$

where $\mathbf{Y}_t = (Y_t(1), \ldots, Y_t(d))^\top$ is a $d \times 1$ random vector ; for fixed ν and predetermined value s, the random vector $\mathbf{Y}_{ns+\nu}$ denotes the realization during the νth season, with $\nu \in \{1, \ldots, s\}$, at year n, $n \in \mathbb{Z}$. The autoregressive model order at season ν is given by $p(\nu)$, and $\mathbf{\Phi}_k(\nu) = (\Phi_{k,ij}(\nu))_{i,j=1,\ldots,d}$, $k = 1, \ldots, p(\nu)$, are the autoregressive model coefficients during season ν, $\nu = 1, \ldots, s$. The error process $\boldsymbol{\epsilon} = \{\boldsymbol{\epsilon}_t, t \in \mathbb{Z}\}$ in (3.1) corresponds to a zero mean periodic white noise, that is $\boldsymbol{\epsilon}$ is composed of independent $d \times 1$ random vectors $\boldsymbol{\epsilon}_t = (\epsilon_t(1), \ldots, \epsilon_t(d))^\top$, such that $E(\boldsymbol{\epsilon}_t) = \mathbf{0}$ and $E(\boldsymbol{\epsilon}_{ns+\nu}\boldsymbol{\epsilon}_{ns+\nu}^\top) = \mathbf{\Sigma}_{\boldsymbol{\epsilon}}(\nu)$, where the error covariance matrix $\mathbf{\Sigma}_{\boldsymbol{\epsilon}}(\nu) = (\sigma_{\boldsymbol{\epsilon},ij}(\nu))_{i,j=1,\ldots,d}$ is assumed to be non singular, $\nu = 1, \ldots, s$. The PVAR process (3.1) is supposed to have a zero mean, that is $E(\mathbf{Y}_t) = \mathbf{0}$. In practical applications, trends and seasonal means are first removed from the series, meaning that a model is formulated by examining $\mathbf{Y}_{ns+\nu} - \boldsymbol{\mu}_\nu$, say, where in general the mathematical expectation $E(\mathbf{Y}_{ns+\nu}) = \boldsymbol{\mu}_\nu$ may be a function of season ν. If $s = 1$, then model (3.1) reduces to a classical vector autoregressive model (VAR).

The model generated by (3.1) is useful for modeling various time series drawn from climatology, hydrology, economics, and electrical engineering, amongst others. See for example Lund and Basawa (1999, 2000), Basawa and Lund (2001), Lund, Shao and Basawa (2006) and the references cited in these papers for applications of periodic time series models to these fields. Historically, the concept of periodically correlated stochastic processes goes back to Gladyshev (1961). >From a statistical point of view, there has been considerable interest in estimating and testing in periodic time series models. Pioneer work in the statistical literature has been done by Jones and Brelsford (1967), Pagano (1978) and Troutman (1979), who have examined the fundamental properties of univariate periodic autoregressive (PAR) processes, the inference techniques for estimating the autoregressive

parameters and for calculating predictions, and the connections with the related stationary multivariate autoregressive process. Parzen and Pagano (1979) have also presented modeling considerations for periodic processes, with empirical applications in economics. Vecchia (1985a, 1985b) has studied maximum likelihood estimation in univariate periodic autoregressive moving average models (PARMA), and has also studied the exact likelihood function for Gaussian processes. Links with vector autoregressive moving average (VARMA) causality results have also been provided, giving sufficient conditions for the existence of a unique periodically correlated solution to a periodic linear difference equation. Vecchia and Ballerini (1991) have presented statistical procedures for deciding if periodicities exist in the autocorrelation function of a seasonal time series. In McLeod (1993), diagnostic test statistics have been proposed for periodic correlation in the residuals of fitted autoregressive moving average models; and in McLeod (1994) we find a complete description of the usual stages of model development in univariate PAR models. Furthermore, having established the asymptotic distribution of the residual autocorrelations he has used this asymptotic result to deduce test statistics of the portmanteau type. He has investigated empirical experiments and found that a modified portmanteau test statistic offers better finite-sample properties than the original version. Lund and Basawa (2000) have explored recursive prediction and likelihood evaluation techniques for PARMA models, and Basawa and Lund (2001) have studied the asymptotic properties of the estimators of the model parameters of causal and invertible PARMA models. Since the number of parameters in periodic time series models can be quite large, Lund, Shao and Basawa (2006) have investigated parsimonious representations in the class of periodic time series models, in order to reduce the number of independent parameters. The results obtained in the papers discussed so far concentrate on univariate periodic time series. However, multivariate models are expected to be more useful in practice, since most real-life situations involve several variables and vector time series. For multivariate periodic time series models, Ula (1990) has studied periodic covariance stationarity conditions for multivariate

PARMA processes, and Ula (1993) has considered the minimum mean square predictor error in that multivariate framework. Franses and Paap (2004) have stated stationarity conditions for a PVAR model with four seasons and an autoregressive order equal to one for each season; they have also discussed how to estimate the parameters of that model. Lütkepohl (2005) has studied maximum likelihood estimation of the model parameters of a general PVAR stochastic process and has discussed test statistics for time invariance of the model coefficients.

In this article, we first give new results on the least squares estimators of the model parameters in the PVAR model defined by (3.1). Since multivariate periodic processes may imply an important number of independent parameters, we consider situations where there are potentially linear constraints on the parameters of a given season. Important special cases include zero-valued parameters on certain components of the autoregressive model parameters. From a model-building point of view, it is well recognized that checking the adequacy of models appears to be a fundamental step in the time series methodology. Residual autocovariances and autocorrelations from classical VARMA models have been found useful for diagnosing a particular fitted model. The monograph of Li (2004) provides an overview of diagnostic checking for time series models using sample autocovariances or autocorrelations. As a second objective, we derive the asymptotic distribution of the residual autocovariance and autocorrelation matrices in the context of PVAR models. Our results generalize the literature in several directions. They extend previous theorems establishing the asymptotic distributions of residual autocovariance and autocorrelation matrices in VAR models with parameter constraints. See Lütkepohl (2005), amongst others. Furthermore, our asymptotic results provide multivariate generalizations of theorems obtained by McLeod (1994) for PAR models. As a useful application of these results, portmanteau test statistics based on a fixed number of residual autocovariance matrices are proposed for diagnosing PVAR models. Our portmanteau test statistics can be calculated for each season, and an omnibus version depending on all seasons can also be considered. The resulting test statistics generalize Hosking's (1980) multivariate portmanteau test statistic designed for VARMA models, and they represent multivariate

versions of test statistics originally proposed by McLeod (1994) and Hipel and McLeod (1994), for diagnosing PAR models. We discuss the asymptotic distributions of these new test statistics and we consider modified versions with better finite-sample properties.

The paper is organized as follows. In Section 3.2, some preliminaries are given, where some basic properties of PVAR models are presented and the sample autocovariance and autocorrelation matrices are defined. Least squares estimators are studied in Section 3.3. In Section 3.4, the asymptotic distributions of the residual autocovariance matrices under the null hypothesis of model adequacy are derived. As a complementary result, we obtain the asymptotic distributions of the residual autocorrelation matrices. We describe applications for diagnostic checking based on these asymptotic results, by introducing portmanteau test statistics. In Section 3.5, some simulation results are reported and Section 3.6 considers an application using the bivariate quarterly West German data studied previously in Lütkepohl (2005). Section 3.7 offers some concluding remarks.

3.2. PRELIMINARIES

3.2.1. Causality, stationarity, and theoretical autocovariance function of PVAR stochastic processes

This section overviews some basic properties of PVAR time series models. In general, the autoregressive orders $p(\nu)$ in model (3.1) may not be constant as a function of ν. However, from a theoretical point of view and in the mathematical developments presented in this section, there is no loss of generality in taking $p(\nu)$ to be constant in ν, by setting :

$$p = \max_{\nu=1,\ldots,s} p(\nu),$$

and imposing the constraints $\mathbf{\Phi}_k(\nu) = \mathbf{0}$ for $p(\nu) < k \leq p$. See, e.g., Troutman (1979) or Lund and Basawa (2000). However, in real applications and from a model-selection perspective, it may be preferable to allow $p(\nu)$ to be different across the seasons; this is illustrated in Section 3.6 with the West German data.

First, we note that equation (3.1) offers a VAR representation :

$$\mathbf{\Phi}_0^* \mathbf{Y}_n^* = \sum_{k=1}^{p^*} \mathbf{\Phi}_k^* \mathbf{Y}_{n-k}^* + \boldsymbol{\epsilon}_n^*, \tag{3.2}$$

where $\mathbf{Y}_n^* = (\mathbf{Y}_{ns+s}^\top, \mathbf{Y}_{ns+s-1}^\top, \ldots, \mathbf{Y}_{ns+1}^\top)^\top$ and $\boldsymbol{\epsilon}_n^* = (\boldsymbol{\epsilon}_{ns+s}^\top, \boldsymbol{\epsilon}_{ns+s-1}^\top, \ldots, \boldsymbol{\epsilon}_{ns+1}^\top)^\top$ are $(ds) \times 1$ random vectors. The autoregressive model order in (3.2) is given by $p^* = \lceil p/s \rceil$, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to the real number x. The matrix $\mathbf{\Phi}_0^*$, and the autoregressive coefficients $\mathbf{\Phi}_k^*$, $k = 1, \ldots, p^*$, all of dimension $(ds) \times (ds)$, are given by the non-singular matrix :

$$\mathbf{\Phi}_0^* = \begin{bmatrix} \mathbf{I}_d & -\mathbf{\Phi}_1(s) & -\mathbf{\Phi}_2(s) & \ldots & -\mathbf{\Phi}_{s-2}(s) & -\mathbf{\Phi}_{s-1}(s) \\ \mathbf{0} & \mathbf{I}_d & -\mathbf{\Phi}_1(s-1) & \ldots & -\mathbf{\Phi}_{s-3}(s-1) & -\mathbf{\Phi}_{s-2}(s-1) \\ \vdots & & & \ddots & & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{I}_d & -\mathbf{\Phi}_1(2) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{I}_d \end{bmatrix},$$

where $\mathbf{I}_d$ denotes the $d \times d$ identity matrix, and :

$$\mathbf{\Phi}_k^* = \begin{bmatrix} \mathbf{\Phi}_{ks}(s) & \mathbf{\Phi}_{ks+1}(s) & \ldots & \mathbf{\Phi}_{ks+s-1}(s) \\ \mathbf{\Phi}_{ks-1}(s-1) & \mathbf{\Phi}_{ks}(s-1) & \ldots & \mathbf{\Phi}_{ks+s-2}(s-1) \\ \vdots & & \ddots & \vdots \\ \mathbf{\Phi}_{ks-s+1}(1) & \mathbf{\Phi}_{ks-s+2}(1) & \ldots & \mathbf{\Phi}_{ks}(1) \end{bmatrix},$$

where $k = 1, 2, \ldots, p^*$ and $\mathbf{\Phi}_k(\nu) = \mathbf{0}$, $k > p$. See also Gladyshev (1961), Pagano (1978), Troutman (1979), Vecchia (1985a, 1985b), Franses and Paap (2004) and Lütkepohl (2005) for developments exploiting the multivariate VAR representation (3.2).

Using general properties of VAR models (see, e.g., Brockwell and Davis (1991)), it follows that the multivariate stochastic process $\{\mathbf{Y}_t^*\}$ is causal if :

$$\det\left(\mathbf{\Phi}_0^* - \mathbf{\Phi}_1^* z - \ldots - \mathbf{\Phi}_{p^*} z^{p^*}\right) \neq 0, \tag{3.3}$$

for all complex numbers z satisfying the condition $|z| \leq 1$, where $\det(\mathbf{A})$ denotes the determinant of the squared matrix $\mathbf{A}$. Equivalently, condition (3.3) can be expressed as :

$$\det\left(\mathbf{I}_{ds} - \mathbf{\Phi}_0^{*-1}\mathbf{\Phi}_1^* z - \ldots - \mathbf{\Phi}_0^{*-1}\mathbf{\Phi}_{p^*} z^{p^*}\right) \neq 0, \quad |z| \leq 1.$$

For example, consider a PVAR process of autoregressive orders $p(\nu) \equiv 1$, $\nu = 1, \ldots, s$. The stationarity condition (3.3) reduces to :

$$\det\left(\mathbf{\Phi}_0^* - \mathbf{\Phi}_1^* z\right) = \det\left[\mathbf{I}_d - \left\{\prod_{k=0}^{s-1} \mathbf{\Phi}_1(s-k)\right\} z\right] \neq 0, \tag{3.4}$$

for all z such that $|z| \leq 1$, and stationarity results for VAR(1) processes can be invoked easily (see, e.g., Brockwell and Davis (1991)). In this special case, it suffices to examine the eigenvalues of $\prod_{k=0}^{s-1} \mathbf{\Phi}_1(s-k)$ and to check if they are all strictly smaller than one in modulus, in order to have stationarity. For a general PVAR process, the stationarity condition rapidly becomes complicated and difficult to detail explicitly.

If process (3.1) corresponds to a causal process, it is possible to represent $\{\mathbf{Y}_{ns+\nu}\}$ through an infinite order moving average expansion :

$$\mathbf{Y}_{ns+\nu} = \sum_{k=0}^{\infty} \mathbf{\Psi}_k(\nu) \boldsymbol{\epsilon}_{ns+\nu-k}, \tag{3.5}$$

where $\mathbf{\Psi}_0(\nu) = \mathbf{I}_d$. The $d \times d$ matrices $\mathbf{\Psi}_k(\nu)$ can be interpreted as seasonal weights, and they satisfy the following conditions :

$$\sum_{k=0}^{\infty} ||\mathbf{\Psi}_k(\nu)|| < \infty, \ \nu = 1, 2, \ldots, s,$$

where $||\mathbf{A}||$ denotes the Euclidian norm of the matrix $\mathbf{A}$, that is

$$||\mathbf{A}|| = \{\mathrm{tr}(\mathbf{A}\mathbf{A}^\top)\}^{1/2},$$

with $\mathrm{tr}(\mathbf{B})$ being the trace of the squared matrix $\mathbf{B}$. The matrices $\mathbf{\Psi}_k(\nu)$ can be found using the recursive relations :

$$\mathbf{\Psi}_k(\nu) = \sum_{j=1}^{\min(k,p)} \mathbf{\Phi}_j(\nu) \mathbf{\Psi}_{k-j}(\nu - j), \ \forall k \geq 1, \ \nu = 1, \ldots, s. \tag{3.6}$$

We note that the notations used in (3.6) and elsewhere interpret $\mathbf{\Psi}_k(j)$, $\forall k \geq 0$, and $\mathbf{\Phi}_k(j)$, $k = 1, \ldots, p$, periodically in j with period s. Interestingly, for $k > p$, the matrix $\mathbf{\Psi}_k(\nu)$ relies solely on the p matrices $\mathbf{\Psi}_{k-1}(\nu-1), \ldots, \mathbf{\Psi}_{k-p}(\nu-p)$, meaning that the recursive relations remain numerically tractable as k becomes larger (Lund and Basawa (2000, p. 77)).

Using the algebric equivalence between multivariate stationarity and periodic correlation (Gladyshev (1961), Ula (1990)), the ds-dimensional process $\{\mathbf{Y}_n^*\}$ is stationary if and only if the d-dimensional process $\{\mathbf{Y}_t\}$ is periodic stationary with period s, in the sense that :

$$\text{cov}(\mathbf{Y}_{n+s}, \mathbf{Y}_{m+s}) = \text{cov}(\mathbf{Y}_n, \mathbf{Y}_m),$$

for all integers n and m. Periodically correlated stochastic processes are also called cyclostationary or periodically stationary (Lund and Basawa (2000)).

The seasonal autocovariance function of the zero-mean process $\{\mathbf{Y}_t\}$ is defined as :

$$\mathbf{\Gamma}_{\mathbf{Y}}(h;\nu) = \text{cov}(\mathbf{Y}_{ns+\nu}, \mathbf{Y}_{ns+\nu-h}) = E(\mathbf{Y}_{ns+\nu}\mathbf{Y}_{ns+\nu-h}^{\top}),$$

which may depend on both lag h and season ν, but not on year n. The autocovariance $\mathbf{\Gamma}_{\mathbf{Y}}(h;\nu)$ is interpreted periodically in ν with period s, using the relations :

$$\begin{aligned}
\mathbf{\Gamma}_{\mathbf{Y}}(h;\nu) &= \text{cov}(\mathbf{Y}_{ns+\nu}, \mathbf{Y}_{ns+\nu-h}), \\
&= \text{cov}(\mathbf{Y}_{(n+1)s+\nu}, \mathbf{Y}_{(n+1)s+\nu-h}), \\
&= \mathbf{\Gamma}_{\mathbf{Y}}(h;\nu+s).
\end{aligned}$$

Similar arguments give the following relation for negative lags :

$$\mathbf{\Gamma}_{\mathbf{Y}}(-h;\nu) = \mathbf{\Gamma}_{\mathbf{Y}}^{\top}(h;\nu+h). \tag{3.7}$$

Using the moving average expression (3.5), it is possible to show that the seasonal autocovariance function satisfies :

$$\mathbf{\Gamma}_{\mathbf{Y}}(h;\nu) = \sum_{k=0}^{\infty} \mathbf{\Psi}_{k+h}(\nu)\mathbf{\Sigma}_{\boldsymbol{\epsilon}}(\nu-k-h)\mathbf{\Psi}_k^{\top}(\nu-h), \tag{3.8}$$

where the covariance matrix $\mathbf{\Sigma}_{\boldsymbol{\epsilon}}(\nu)$ is interpreted periodically in ν with period s. Multiplying (3.1) by $\mathbf{Y}_{ns+\nu-h}^{\top}$ for $h = 0, 1, \ldots, p$ and taking expectations give a periodic version of the Yule-Walker equations. See also Lund and Basawa (2000, pp. 91-92) in the univariate case. As for VAR stochastic processes, the autocovariance function of a PVAR process can be calculated recursively :

$$\begin{aligned}
\mathbf{\Gamma}_{\mathbf{Y}}(h;\nu) &= \mathbf{\Phi}_1(\nu)\mathbf{\Gamma}_{\mathbf{Y}}(h-1;\nu-1) + \mathbf{\Phi}_2(\nu)\mathbf{\Gamma}_{\mathbf{Y}}(h-2;\nu-2) + \ldots + \\
&\quad \mathbf{\Phi}_p(\nu)\mathbf{\Gamma}_{\mathbf{Y}}(h-p;\nu-p).
\end{aligned} \tag{3.9}$$

Once the theoretical autocovariances $\mathbf{\Gamma}_{\mathbf{Y}}(h;\nu)$ are determined for $0 \leq h \leq p$ and seasons $\nu = 1, 2, \ldots, s$, the autocovariances for lags $h > p$ can be uniquely solved using the recursive relation (3.9).

3.2.2. Sample autocovariances and autocorrelations

Let $\boldsymbol{\beta}(\nu) = (\text{vec}^\top\{\mathbf{\Phi}_1(\nu)\}, \ldots, \text{vec}^\top\{\mathbf{\Phi}_{p(\nu)}(\nu)\})^\top$ be a $\{d^2p(\nu)\} \times 1$ vector of parameters, where vec($\mathbf{A}$) corresponds to the vector obtained by stacking the columns of $\mathbf{A}$ (see Harville (1997, Chapter 16.3)). We assume that, for a known $\{d^2p(\nu)\} \times K(\nu)$ matrix $\mathbf{R}(\nu)$ of rank $K(\nu)$, and a known $\{d^2p(\nu)\} \times 1$ vector $\mathbf{b}(\nu)$, the following relation is satisfied :

$$\boldsymbol{\beta}(\nu) = \mathbf{R}(\nu)\boldsymbol{\xi}(\nu) + \mathbf{b}(\nu), \tag{3.10}$$

where $\boldsymbol{\xi}(\nu)$ represents a $K(\nu) \times 1$ vector of unknown parameters.

Letting $\mathbf{R}(\nu) = \mathbf{I}_{d^2p(\nu)}$, $\mathbf{b}(\nu) = \mathbf{0}$, $\nu = 1, \ldots, s$ give what we call the full unconstrained case. In general, the matrices $\mathbf{R}(\nu)$ and the vectors $\mathbf{b}(\nu)$ allow for linear constraints on the parameters of the same season ν, $\nu = 1, \ldots, s$.

For any

$$\dot{\boldsymbol{\beta}}(\nu) = \mathbf{R}(\nu)\dot{\boldsymbol{\xi}}(\nu) + \mathbf{b}(\nu),$$

where $\dot{\boldsymbol{\beta}}(\nu) = (\text{vec}^\top\{\dot{\mathbf{\Phi}}_1(\nu)\}, \ldots, \text{vec}^\top\{\dot{\mathbf{\Phi}}_{p(\nu)}(\nu)\})^\top$, with general $d \times d$ matrices $\dot{\mathbf{\Phi}}_k(\nu)$, $k = 1, \ldots, p(\nu)$, we introduce the model residuals :

$$\dot{\boldsymbol{\epsilon}}_{ns+\nu} = \begin{cases} \mathbf{Y}_{ns+\nu} - \sum_{k=1}^{p(\nu)} \dot{\mathbf{\Phi}}_k(\nu)\mathbf{Y}_{ns+\nu-k}, & ns+\nu > p(\nu), \\ \mathbf{0}, & ns+\nu \leq p(\nu), \end{cases}$$

which are well-defined for $n = 0, 1, \ldots, N-1$. As in Li and McLeod (1981), we use the dot notation to designate the residuals $\dot{\boldsymbol{\epsilon}}_{ns+\nu}$, $n = 0, 1, \ldots, N-1$, expressed in function of the general quantities $\dot{\boldsymbol{\beta}}(\nu)$ and $\dot{\boldsymbol{\xi}}(\nu)$, $\nu = 1, \ldots, s$. Using the well-known relation vec($\mathbf{ABC}$) = $(\mathbf{C}^\top \otimes \mathbf{A})$vec($\mathbf{B}$), where '$\otimes$' represents the Kronecker product, we obtain for $ns + \nu > p(\nu)$ the relation

$$\dot{\boldsymbol{\epsilon}}_{ns+\nu} = \mathbf{Y}_{ns+\nu} - \sum_{k=1}^{p(\nu)} (\mathbf{Y}_{ns+\nu-k}^\top \otimes \mathbf{I}_d)\text{vec}\{\dot{\mathbf{\Phi}}_k(\nu)\}.$$

Let $\mathbf{\Gamma}_{\boldsymbol{\epsilon}}(h;\nu) = \mathrm{cov}(\boldsymbol{\epsilon}_{ns+\nu}, \boldsymbol{\epsilon}_{ns+\nu-h})$ be the lag h theoretical autocovariance matrix at season ν of the error process $\boldsymbol{\epsilon}$. Let

$$\boldsymbol{\gamma}_{\boldsymbol{\epsilon}}(\nu) = \left(\boldsymbol{\gamma}_{\boldsymbol{\epsilon}}^{\top}(1;\nu), \boldsymbol{\gamma}_{\boldsymbol{\epsilon}}^{\top}(2;\nu), \ldots, \boldsymbol{\gamma}_{\boldsymbol{\epsilon}}^{\top}(M;\nu)\right)^{\top} \tag{3.11}$$

be a $(d^2M) \times 1$ vector of theoretical autocovariances, where $\boldsymbol{\gamma}_{\boldsymbol{\epsilon}}(h;\nu) = \mathrm{vec}\{\mathbf{\Gamma}_{\boldsymbol{\epsilon}}(h;\nu)\}$. We define $\boldsymbol{\rho}_{\boldsymbol{\epsilon}}(h) = \mathbf{\Gamma}_0^{-1}(\nu)\mathbf{\Gamma}_{\boldsymbol{\epsilon}}(h;\nu)\mathbf{\Gamma}_0^{-1}(\nu-h)$ the lag h theoretical autocorrelation matrix at season ν, using the diagonal matrix $\mathbf{\Gamma}_0(\nu) = \mathrm{diag}(\boldsymbol{\sigma}_{\boldsymbol{\epsilon},11}^{1/2}(\nu), \ldots, \boldsymbol{\sigma}_{\boldsymbol{\epsilon},dd}^{1/2}(\nu))$. We introduce the sample autocovariance matrices $\mathbf{C}_{\dot{\boldsymbol{\epsilon}}}(h;\nu) = \left(C_{\dot{\boldsymbol{\epsilon}},ij}(h;\nu)\right)_{i,j=1,\ldots,d}$:

$$\mathbf{C}_{\dot{\boldsymbol{\epsilon}}}(h;\nu) = \begin{cases} N^{-1}\sum_{n=h}^{N-1} \dot{\boldsymbol{\epsilon}}_{ns+\nu}\dot{\boldsymbol{\epsilon}}_{ns+\nu-h}^{\top}, & h \geq 0, \\ \mathbf{C}_{\dot{\boldsymbol{\epsilon}}}^{\top}(-h;\nu-h), & h < 0. \end{cases}$$

Let

$$\mathbf{c}_{\dot{\boldsymbol{\epsilon}}}(\nu) = (\mathbf{c}_{\dot{\boldsymbol{\epsilon}}}^{\top}(1;\nu), \ldots, \mathbf{c}_{\dot{\boldsymbol{\epsilon}}}^{\top}(M;\nu))^{\top} \tag{3.12}$$

be the $(d^2M)\times 1$ vector of sample autocovariances, where $\mathbf{c}_{\dot{\boldsymbol{\epsilon}}}(h;\nu) = \mathrm{vec}\{\mathbf{C}_{\dot{\boldsymbol{\epsilon}}}(h;\nu)\}$. Here M represents a fixed integer with respect to the sample size $n = Ns$, satisfying the relation $1 \leq M < N$; this constant is the maximal lag order. Similarly, the vector of sample autocorrelations is given by $\mathbf{r}_{\dot{\boldsymbol{\epsilon}}}(\nu) = (\mathbf{r}_{\dot{\boldsymbol{\epsilon}}}^{\top}(1;\nu), \ldots, \mathbf{r}_{\dot{\boldsymbol{\epsilon}}}^{\top}(M;\nu))^{\top}$, where :

$$\begin{aligned} \mathbf{r}_{\dot{\boldsymbol{\epsilon}}}(h;\nu) &= \mathrm{vec}\left\{\mathbf{D}_{\dot{\boldsymbol{\epsilon}}}^{-1}(\nu)\mathbf{C}_{\dot{\boldsymbol{\epsilon}}}(h;\nu)\mathbf{D}_{\dot{\boldsymbol{\epsilon}}}^{-1}(\nu-h)\right\}, \\ &= \left(\mathbf{D}_{\dot{\boldsymbol{\epsilon}}}^{-1}(\nu-h) \otimes \mathbf{D}_{\dot{\boldsymbol{\epsilon}}}^{-1}(\nu)\right)\mathbf{c}_{\dot{\boldsymbol{\epsilon}}}(h;\nu), \end{aligned}$$

with $\mathbf{D}_{\dot{\boldsymbol{\epsilon}}}(\nu) = \mathrm{diag}\left(\mathbf{C}_{\dot{\boldsymbol{\epsilon}},11}^{1/2}(0;\nu), \ldots, \mathbf{C}_{\dot{\boldsymbol{\epsilon}},dd}^{1/2}(0;\nu)\right)$.

In Sections 3.3 and 3.4, we discuss the asymptotic properties of the least squares estimators in the unconstrained and constrained cases, and we establish the asymptotic distributions of the residual autocovariance and autocorrelation vectors $\mathbf{c}_{\hat{\boldsymbol{\epsilon}}}(h;\nu)$ and $\mathbf{r}_{\hat{\boldsymbol{\epsilon}}}(h;\nu)$, $h = 1, \ldots, M$, where $\hat{\boldsymbol{\epsilon}}_{ns+\nu}$, $n = 0, 1, \ldots, N-1$, denote the least squares residuals, $\nu = 1, \ldots, s$.

3.3. UNCONSTRAINED LEAST SQUARES ESTIMATORS AND LEAST SQUARES ESTIMATORS WITH LINEAR CONSTRAINTS ON THE PARAMETERS

In this section, we study the asymptotic properties of least squares estimators from a causal PVAR model. In principle, the asymptotic properties of the PVAR parameter estimators could be deduced from results for multivariate time series, using the general multivariate representation (3.2). See, e.g., Brockwell and Davis (1991), Fuller (1996), Reinsel (1997) or Lütkepohl (2005), amongst others. However, in order to derive the statistical properties of the $p(\nu)$ autoregressive coefficient matrices, for each season ν, $\nu = 1, \ldots, s$, of our PVAR model, the multivariate process (3.2) needs to be inverted; this inversion appears unnecessarily complicated in our multivariate framework. Furthermore, from (3.2), the multivariate VARMA process is not written in a standard VARMA form; the process must be pre-multiplied on each side by the matrix $\mathbf{\Phi}_0^{*-1}$. This rescaling operation complicates the interpretation of the estimated parameters and the derivation of their statistical properties in the original scale, since the covariance matrix of the error term of the standard VARMA model now depends on the autoregressive parameters. These considerations also occur in the univariate setting, see Basawa and Lund (2001). Therefore, it is more informative to work directly with the individual PVAR components. From a practical point of view, the results presented in this section give directly the asymptotic properties of the autoregressive estimators in the original scale, which is also the natural one.

The PVAR model in (3.1) has $d^2 \sum_{\nu=1}^{s} p(\nu)$ autoregressive parameters $\mathbf{\Phi}_k(\nu)$, $k = 1, \ldots, p(\nu)$, $\nu = 1, \ldots, s$, and s additional $d \times d$ covariance matrices $\mathbf{\Sigma}_{\boldsymbol{\epsilon}}(\nu)$, $\nu = 1, \ldots, s$. For multivariate processes, the number of parameters can be quite large; for vector periodic processes, the inflation of parameters is due to the s seasons. For example, in the case of bivariate monthly data where $d = 2$, $s = 12$, and, in the simplest case $p(\nu) \equiv 1$, this means that 48 independent autoregressive parameters must be estimated (by comparison, a traditional VAR(1) process relies on four independent parameters). In view of these considerations, we consider estimation in the unrestricted case but also in the situation where the parameters

of the same season ν satisfy the relation (3.10). This linear constraint includes the important special case of parameters set to zero on certain components of $\mathbf{\Phi}_k(\nu)$, $\nu = 1, \ldots, s$. In practice, a two-step procedure could consist of fitting a full unconstrained model, and, in a second stage of inference, the estimators which are statistically not significant could be considered known zero parameters, providing frequently more parsimonious models.

Consider the time series data $\mathbf{Y}_{ns+\nu}$, $n = 0, 1, \ldots, N-1$, $\nu = 1, \ldots, s$, giving a sample size equal to $n = Ns$. Let

$$\mathbf{Z}(\nu) = \left(\mathbf{Y}_\nu, \mathbf{Y}_{s+\nu}, \ldots, \mathbf{Y}_{(N-1)s+\nu}\right), \tag{3.13}$$

$$\mathbf{E}(\nu) = \left(\boldsymbol{\epsilon}_\nu, \boldsymbol{\epsilon}_{s+\nu}, \ldots, \boldsymbol{\epsilon}_{(N-1)s+\nu}\right), \tag{3.14}$$

$$\mathbf{X}(\nu) = \left(\mathbf{X}_0(\nu), \ldots, \mathbf{X}_{N-1}(\nu)\right), \tag{3.15}$$

be $d \times N$, $d \times N$ and $\{dp(\nu)\} \times N$ random matrices, where

$$\mathbf{X}_n(\nu) = (\mathbf{Y}^\top_{ns+\nu-1}, \ldots, \mathbf{Y}^\top_{ns+\nu-p(\nu)})^\top,$$

$n = 0, 1, \ldots, N-1$, denote $\{dp(\nu)\} \times 1$ random vectors. The PVAR model can be reformulated as :

$$\mathbf{Z}(\nu) = \mathbf{B}(\nu)\mathbf{X}(\nu) + \mathbf{E}(\nu), \ \nu = 1, \ldots, s, \tag{3.16}$$

where the model parameters are collected in the $d \times \{dp(\nu)\}$ matrix $\mathbf{B}(\nu)$ which is defined as :

$$\mathbf{B}(\nu) = \left(\mathbf{\Phi}_1(\nu), \ldots, \mathbf{\Phi}_{p(\nu)}(\nu)\right). \tag{3.17}$$

Vectorizing, we obtain :

$$\begin{aligned}
\mathbf{z}(\nu) &= \{\mathbf{X}^\top(\nu) \otimes \mathbf{I}_d\}\mathrm{vec}\{\mathbf{B}(\nu)\} + \mathrm{vec}\{\mathbf{E}(\nu)\}, \\
&= \{\mathbf{X}^\top(\nu) \otimes \mathbf{I}_d\}\boldsymbol{\beta}(\nu) + \mathbf{e}(\nu), \\
&= \{\mathbf{X}^\top(\nu) \otimes \mathbf{I}_d\}\{\mathbf{R}(\nu)\boldsymbol{\xi}(\nu) + \mathbf{b}(\nu)\} + \mathbf{e}(\nu),
\end{aligned} \tag{3.18}$$

where $\mathbf{z}(\nu) = \mathrm{vec}\{\mathbf{Z}(\nu)\}$, $\boldsymbol{\beta}(\nu) = \mathrm{vec}\{\mathbf{B}(\nu)\}$ and $\mathbf{e}(\nu) = \mathrm{vec}\{\mathbf{E}(\nu)\}$. The covariance matrix of the random vector $\mathbf{e}(\nu)$ is $\mathbf{I}_N \otimes \mathbf{\Sigma}_{\boldsymbol{\epsilon}}(\nu)$.

The least squares estimators of $\boldsymbol{\xi}(\nu)$, $\nu = 1, \ldots, s$ are obtained by minimizing the generalized least squares criterion :

$$S_G(\boldsymbol{\xi}) = \sum_{\nu=1}^{s} \mathbf{e}^\top(\nu)\{\mathbf{I}_N \otimes \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu)\}^{-1}\mathbf{e}(\nu), \tag{3.19}$$

where $\boldsymbol{\xi} = (\boldsymbol{\xi}^\top(1), \ldots, \boldsymbol{\xi}^\top(s))^\top$ represents a $\{\sum_{\nu=1}^{s} K(\nu)\} \times 1$ vector. In the next subsections, we discuss separately the unrestricted and restricted cases.

3.3.1. Unconstrained least squares estimators

The least squares estimators based on the generalized least squares criteria are obtained equivalently by minimizing the ordinary least squares (a similar result holds for VAR models, see Lütkepohl (2005, p. 71)) :

$$S(\boldsymbol{\beta}) = \sum_{\nu=1}^{s} \mathbf{e}^\top(\nu)\mathbf{e}(\nu), \tag{3.20}$$

where $\boldsymbol{\beta} = (\boldsymbol{\beta}^\top(1), \ldots, \boldsymbol{\beta}^\top(s))^\top$ is the $\{d^2 \sum_{\nu=1}^{s} p(\nu)\} \times 1$ vector of model parameters. To obtain the least squares estimators, we differentiate $S(\boldsymbol{\beta})$ with respect to each parameter $\boldsymbol{\Phi}_k(\nu)$, $k = 1, \ldots, p(\nu)$, $\nu = 1, \ldots, s$. Thus we obtain easily :

$$\frac{\partial S(\boldsymbol{\beta})}{\partial \mathrm{vec}\{\boldsymbol{\Phi}_k(\nu)\}} = -2\sum_{n=0}^{N-1}(\mathbf{Y}_{ns+\nu-k} \otimes \boldsymbol{\epsilon}_{ns+\nu}),\ k = 1, \ldots, p(\nu),\ \nu = 1, \ldots, s.$$

Setting the derivatives equal to zero, $k = 1, \ldots, p(\nu)$, gives the following system for a given season ν :

$$\sum_{n=0}^{N-1}\{\mathbf{X}_n(\nu) \otimes \boldsymbol{\epsilon}_{ns+\nu}\} = \mathbf{0},$$

where $\mathbf{0}$ is the $\{d^2 p(\nu)\} \times 1$ null vector. Since $\boldsymbol{\epsilon}_{ns+\nu} = \mathbf{Y}_{ns+\nu} - \{\mathbf{X}_n^\top(\nu) \otimes \mathbf{I}_d\}\boldsymbol{\beta}(\nu)$, the normal equations at season ν are :

$$\sum_{n=0}^{N-1}\{\mathbf{X}_n(\nu) \otimes \mathbf{Y}_{ns+\nu}\} = \left[\sum_{n=0}^{N-1}\left\{\mathbf{X}_n(\nu)\mathbf{X}_n^\top(\nu) \otimes \mathbf{I}_d\right\}\right]\boldsymbol{\beta}(\nu).$$

Consequently, the least squares estimators of $\boldsymbol{\beta}(\nu)$ satisfy the relation :

$$\hat{\boldsymbol{\beta}}(\nu) = \left[\{\mathbf{X}(\nu)\mathbf{X}^\top(\nu)\}^{-1}\mathbf{X}(\nu) \otimes \mathbf{I}_d\right]\mathbf{z}(\nu),$$

and the residuals are $\hat{\boldsymbol{\epsilon}}_{ns+\nu} = \mathbf{Y}_{ns+\nu} - \{\mathbf{X}_n^\top(\nu) \otimes \mathbf{I}_d\}\hat{\boldsymbol{\beta}}(\nu)$. Using the properties of the vec$(\cdot)$ operator, it should be noted that an alternative expression for the least squares estimators is given by :

$$\hat{\mathbf{B}}(\nu) = \mathbf{Z}(\nu)\mathbf{X}^\top(\nu)\{\mathbf{X}(\nu)\mathbf{X}^\top(\nu)\}^{-1}. \tag{3.21}$$

The asymptotic properties of the least squares estimators in the unrestricted case are stated in Theorem 3.1. The symbols '$\xrightarrow{d}$' and '$\xrightarrow{p}$' stand for convergence in distribution and probability, respectively, and $N_d(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ denotes a d-dimensional normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$.

Theorem 3.1. *Let a time series be generated by equation (3.1) and presume that the causality condition given by (3.3) is satisfied. Suppose that the error term* $\boldsymbol{\epsilon} = \{\boldsymbol{\epsilon}_t\}$ *corresponds to a periodic white noise composed of independent* $d \times 1$ *random vectors satisfying* $E(\boldsymbol{\epsilon}_{ns+\nu}) = \mathbf{0}$, *and* $\mathrm{var}(\boldsymbol{\epsilon}_{ns+\nu}) = \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu)$, $\nu = 1, \ldots, s$. *Suppose that the fourth-order moments of* $\boldsymbol{\epsilon}_t$ *are finite :*

$$E\{|\epsilon_t(i)\epsilon_t(j)\epsilon_t(k)\epsilon_t(l)|\} < \infty, \ \forall i, j, k, l = 1, \ldots, d; \ \forall t.$$

Then for $\nu = 1, \ldots, s$ *:*

$$N^{-1/2} \sum_{n=0}^{N-1} \mathrm{vec}\{\boldsymbol{\epsilon}_{ns+\nu}\mathbf{X}_n^\top(\nu)\} \xrightarrow{d} N_{d^2p(\nu)}\left(\mathbf{0}, \boldsymbol{\Omega}(\nu) \otimes \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu)\right), \tag{3.22}$$

$$\hat{\boldsymbol{\beta}}(\nu) \xrightarrow{p} \boldsymbol{\beta}(\nu), \tag{3.23}$$

$$N^{1/2}\{\hat{\boldsymbol{\beta}}(\nu) - \boldsymbol{\beta}(\nu)\} \xrightarrow{d} N_{d^2p(\nu)}\left(\mathbf{0}, \boldsymbol{\Omega}^{-1}(\nu) \otimes \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu)\right), \tag{3.24}$$

where $\boldsymbol{\Omega}(\nu)$ *corresponds to the* $\{dp(\nu)\} \times \{dp(\nu)\}$ *covariance matrix of the* $\{dp(\nu)\} \times 1$ *random vector* $\mathbf{X}_n(\nu)$. *Furthermore,* $N^{1/2}\{\hat{\boldsymbol{\beta}}(\nu) - \boldsymbol{\beta}(\nu)\}$ *and* $N^{1/2}\{\hat{\boldsymbol{\beta}}(\nu') - \boldsymbol{\beta}(\nu')\}$ *are asymptotically independent,* $\nu \neq \nu'$, $\nu, \nu' = 1, \ldots, s$.

PROOF. Let $\mathcal{F}_{n-1} = \sigma(\mathbf{Y}_{ns}, \mathbf{Y}_{ns-1}, \ldots)$ be the sigma-algebra associated with the random vectors $\{\mathbf{Y}_{ns-k}, k \geq 0\}$. The independence assumption of the error term $\{\boldsymbol{\epsilon}_t\}$ allows us to show that $\{\mathrm{vec}\{\boldsymbol{\epsilon}_{ns+\nu}\mathbf{X}_n^\top(\nu)\}\}$ is a martingale difference sequence, that is $E[\mathrm{vec}\{\boldsymbol{\epsilon}_{ns+\nu}\mathbf{X}_n^\top(\nu)\}|\mathcal{F}_{n-1}] = \mathbf{0}$, where $\mathbf{0}$ is the $\{d^2p(\nu)\} \times 1$ null vector. Furthermore, the unconditional covariance matrix of the random vector $\mathrm{vec}\{\boldsymbol{\epsilon}_{ns+\nu}\mathbf{X}_n^\top(\nu)\}$ is given by $\boldsymbol{\Omega}(\nu) \otimes \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu)$. Invoking the law of large numbers

for martingale difference sequences (see, e.g., Hamilton (1994, Chap. 7) or White (2001, Chap. 3)), it follows that :

$$N^{-1}\sum_{n=0}^{N-1}\mathrm{vec}\{\boldsymbol{\epsilon}_{ns+\nu}\mathbf{X}_n^\top(\nu)\} \xrightarrow{p} \mathbf{0},$$

where the dimension of $\mathbf{0}$ is $\{d^2p(\nu)\}\times 1$, and

$$N^{-1}\sum_{n=0}^{N-1}\mathrm{vec}\{\boldsymbol{\epsilon}_{ns+\nu}\mathbf{X}_n^\top(\nu)\}\mathrm{vec}^\top\{\boldsymbol{\epsilon}_{ns+\nu}\mathbf{X}_n^\top(\nu)\} \xrightarrow{p} \boldsymbol{\Omega}(\nu)\otimes\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu).$$

Using a central limit theorem for martingale difference sequences (see, e.g., Hamilton (1994, Chap. 7) or White (2001, Chap. 5)), we deduce (3.22).

Using the relation (3.21), we can write :

$$\hat{\mathbf{B}}(\nu)-\mathbf{B}(\nu)=N^{-1}\mathbf{E}(\nu)\mathbf{X}^\top(\nu)\{N^{-1}\mathbf{X}(\nu)\mathbf{X}^\top(\nu)\}^{-1}.$$

Noting that

$$\sum_{n=0}^{N-1}\mathrm{vec}\{\boldsymbol{\epsilon}_{ns+\nu}\mathbf{X}_n^\top(\nu)\}=\mathrm{vec}\{\mathbf{E}(\nu)\mathbf{X}^\top(\nu)\},$$

it follows that

$$N^{-1/2}\mathrm{vec}\{\mathbf{E}(\nu)\mathbf{X}^\top(\nu)\} \xrightarrow{d} N_{d^2p(\nu)}(\mathbf{0},\boldsymbol{\Omega}(\nu)\otimes\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu)),$$

$$N^{-1}\mathrm{vec}\{\mathbf{E}(\nu)\mathbf{X}^\top(\nu)\} \xrightarrow{p} \mathbf{0},$$

where the dimension of $\mathbf{0}$ is $\{d^2p(\nu)\}\times 1$, and also $\{N^{-1}\mathbf{X}(\nu)\mathbf{X}^\top(\nu)\}^{-1} \xrightarrow{p} \boldsymbol{\Omega}^{-1}(\nu)$; these results show (3.23). Since

$$N^{1/2}\{\hat{\boldsymbol{\beta}}(\nu)-\boldsymbol{\beta}(\nu)\}=\left[\{N^{-1}\mathbf{X}(\nu)\mathbf{X}^\top(\nu)\}^{-1}\otimes\mathbf{I}_d\right]N^{-1/2}\{\mathbf{X}(\nu)\otimes\mathbf{I}_d\}\mathbf{e}(\nu), \quad (3.25)$$

Slutsky's theorem and relation (3.22) give (3.24), using the following argument :

$$\{\boldsymbol{\Omega}^{-1}(\nu)\otimes\mathbf{I}_d\}\{\boldsymbol{\Omega}(\nu)\otimes\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu)\}\{\boldsymbol{\Omega}^{-1}(\nu)\otimes\mathbf{I}_d\}=\{\boldsymbol{\Omega}^{-1}(\nu)\otimes\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu)\}.$$

The joint asymptotic normality of $N^{1/2}\{\hat{\boldsymbol{\beta}}(1)-\boldsymbol{\beta}(1),\ldots,\hat{\boldsymbol{\beta}}(s)-\boldsymbol{\beta}(s)\}$ follows using the same kind of manipulations as those for a single season ν, and from this the asymptotic independence between $N^{1/2}\{\hat{\boldsymbol{\beta}}(\nu)-\boldsymbol{\beta}(\nu)\}$ and $N^{1/2}\{\hat{\boldsymbol{\beta}}(\nu')-\boldsymbol{\beta}(\nu')\}$, $\nu\neq\nu'$ is easily deduced. □

Theorem 3.1 generalizes to PVAR time series models the univariate results of Pagano (1978) and Vecchia (1985b), under conditions similar to those used in establishing Theorem 3.1 of Basawa and Lund (2001). Lütkepohl (2005, p. 596) studied maximum likelihood estimation of PVAR models, which is, under Gaussian assumptions, equivalent to least square estimation. He also investigated estimation of periodic models under various types of restrictions, including the situations where all the coefficients are time varying and also time invariant. Likelihood ratio (LR) test statistics were obtained, based on the comparison of the log-likelihood functions under these restrictions. However, the asymptotic covariance matrix of the least squares estimators was not explicitly provided; that expression was not necessary to develop LR-test statistics. Here, our asymptotic result gives explicitly the asymptotic covariance matrix of the least squares estimators, which will be useful in studying the approximative distributions of the portmanteau test statistics in Section 3.4.

3.3.2. Least squares estimators with linear constraints

When the parameters satisfy the linear constraint (3.10), the least squares estimators of $\boldsymbol{\xi}(\nu)$, $\nu = 1, \ldots, s$, minimize the generalized criterion (3.19), which is not equivalent to (3.20), see Lütkepohl (2005), amongst others. Recall that from (3.18) we have the following relation :

$$\mathbf{e}(\nu) = \mathbf{z}(\nu) - \{\mathbf{X}^\top(\nu) \otimes \mathbf{I}_d\}\{\mathbf{R}(\nu)\boldsymbol{\xi}(\nu) + \mathbf{b}(\nu)\},$$

which is convenient to derive the asymptotic properties of the least squares estimator of $\boldsymbol{\xi}(\nu)$.

Proceeding as in the previous section, it is possible to show that the least squares estimator $\hat{\boldsymbol{\xi}}(\nu)$ of $\boldsymbol{\xi}(\nu)$ is given by :

$$\begin{aligned}\hat{\boldsymbol{\xi}}(\nu) &= \left[\mathbf{R}^\top(\nu)\{\mathbf{X}(\nu)\mathbf{X}^\top(\nu) \otimes \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\nu)\}\mathbf{R}(\nu)\right]^{-1}\mathbf{R}^\top(\nu)\{\mathbf{X}(\nu) \otimes \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\nu)\} \times \\ &\quad \left[\mathbf{z}(\nu) - \{\mathbf{X}^\top(\nu) \otimes \mathbf{I}_d\}\mathbf{b}(\nu)\right].\end{aligned}$$

Furthermore, the following relation is satisfied :

$$N^{1/2}\{\hat{\boldsymbol{\xi}}(\nu)-\boldsymbol{\xi}(\nu)\} = N^{1/2}\left[\mathbf{R}^\top(\nu)\{\mathbf{X}(\nu)\mathbf{X}^\top(\nu)\otimes\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\nu)\}\mathbf{R}(\nu)\right]^{-1}\mathbf{R}^\top(\nu)\times \{\mathbf{X}(\nu)\otimes\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\nu)\}\mathbf{e}(\nu).$$

Consequently, under the conditions of Theorem 3.1, the estimator $\hat{\boldsymbol{\xi}}(\nu)$ is consistent for $\boldsymbol{\xi}(\nu)$, and $\hat{\boldsymbol{\xi}}(\nu)$ follows asymptotically a normal distribution, that is :

$$N^{1/2}\{\hat{\boldsymbol{\xi}}(\nu)-\boldsymbol{\xi}(\nu)\}\xrightarrow{d} N_{K(\nu)}\left(\mathbf{0},\left[\mathbf{R}^\top(\nu)\{\boldsymbol{\Omega}(\nu)\otimes\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\nu)\}\mathbf{R}(\nu)\right]^{-1}\right).$$

It should be noted that the estimator $\hat{\boldsymbol{\xi}}(\nu)$ is unfeasible in practice, since it relies on the unknown matrix $\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu)$. A feasible estimator is given by :

$$\hat{\hat{\boldsymbol{\xi}}}(\nu) = \left[\mathbf{R}^\top(\nu)\{\mathbf{X}(\nu)\mathbf{X}^\top(\nu)\otimes\tilde{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon}}^{-1}(\nu)\}\mathbf{R}(\nu)\right]^{-1}\mathbf{R}(\nu)\{\mathbf{X}(\nu)\otimes\tilde{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon}}^{-1}(\nu)\}\times [\mathbf{z}(\nu)-\{\mathbf{X}^\top(\nu)\otimes\mathbf{I}_d\}\mathbf{b}(\nu)],$$

where $\tilde{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon}}(\nu)$ denotes a consistent estimator of the covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu)$ for $\nu = 1,\ldots,s$. A possible candidate is obtained from the unconstrained least squares estimators :

$$\tilde{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon}}(\nu)=\{N-dp(\nu)\}^{-1}\left\{\mathbf{Z}(\nu)-\hat{\mathbf{B}}(\nu)\mathbf{X}(\nu)\right\}\left\{\mathbf{Z}(\nu)-\hat{\mathbf{B}}(\nu)\mathbf{X}(\nu)\right\}^\top,$$

where $\hat{\mathbf{B}}(\nu)$ represents the unconstrained least squares estimators (3.21) obtained in Section 3.3.1. The resulting estimator of $\boldsymbol{\beta}(\nu)$ is given by $\hat{\hat{\boldsymbol{\beta}}}(\nu)=\mathbf{R}(\nu)\hat{\hat{\boldsymbol{\xi}}}(\nu)+\mathbf{b}(\nu)$, and its asymptotic distribution is normal :

$$N^{1/2}\{\hat{\hat{\boldsymbol{\beta}}}(\nu)-\boldsymbol{\beta}(\nu)\}\xrightarrow{d} N_{d^2p(\nu)}\left(\mathbf{0},\mathbf{R}(\nu)\left[\mathbf{R}^\top(\nu)\{\boldsymbol{\Omega}(\nu)\otimes\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\nu)\}\mathbf{R}(\nu)\right]^{-1}\mathbf{R}^\top(\nu)\right).$$

The proof of the above result follows, using arguments similar to those called upon by Lütkepohl (2005). More precisely, Lütkepohl (2005, Section 5 and Section 10.3) established the asymptotic properties of least squares estimators in VAR time series models, and in VAR models with stationary exogenous variables, when the model parameters satisfy linear constraints. This section generalizes these results to PVAR models.

3.4. ASYMPTOTIC DISTRIBUTION OF THE RESIDUAL AUTOCOVARIANCE AND AUTOCORRELATION MATRICES

For the class of VAR models with parameter constraints, Lütkepohl (2005) derives the asymptotic distribution of the residual autocovariance and autocorrelation matrices, which follow normal distributions. Duchesne (2005) extends these results in the class of VAR models with stationary exogenous variables and linear constraints. In this section, we establish these kinds of results for PVAR models with linear constraints on the parameters of a given season. From these results, we can propose new test statistics for checking the adequacy of a particular PVAR model, generalizing the proposals of McLeod (1994) and Hipel and McLeod (1994) for diagnosing univariate periodic time series models.

To establish the asymptotic distribution of $N^{1/2}\mathbf{c}_{\hat{\boldsymbol{\epsilon}}}(\nu)$, we first note that :

$$\begin{aligned}\mathbf{c}_{\boldsymbol{\epsilon}}(l;\nu) &= \text{vec}\left\{N^{-1}\sum_{n=l}^{N-1}(\boldsymbol{\epsilon}_{ns+\nu}\boldsymbol{\epsilon}_{ns+\nu-l}^{\top})\right\}, \\ &= N^{-1}\sum_{n=l}^{N-1}(\boldsymbol{\epsilon}_{ns+\nu-l}\otimes\boldsymbol{\epsilon}_{ns+\nu}), \end{aligned} \tag{3.26}$$

where we used the fact that $\text{vec}(\mathbf{ab}^{\top}) = \mathbf{b}\otimes\mathbf{a}$, where $\mathbf{a}$ and $\mathbf{b}$ are two arbitrary vectors. Since $\{\boldsymbol{\epsilon}_t\}$ represents a periodic white noise, it follows that $E\{\mathbf{c}_{\boldsymbol{\epsilon}}(l;\nu)\} = \mathbf{0}$ with covariance matrix :

$$\begin{aligned}\text{var}\{\mathbf{c}_{\boldsymbol{\epsilon}}(l;\nu)\} &= E\{\mathbf{c}_{\boldsymbol{\epsilon}}(l;\nu)\mathbf{c}_{\boldsymbol{\epsilon}}^{\top}(l;\nu)\}, \\ &= N^{-2}(N-l)\{\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu-l)\otimes\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu)\},\end{aligned}$$

since $E(\boldsymbol{\epsilon}_{ns+\nu-l}\boldsymbol{\epsilon}_{ns+\nu-l}^{\top})\otimes E(\boldsymbol{\epsilon}_{ns+\nu}\boldsymbol{\epsilon}_{ns+\nu}^{\top}) = \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu-l)\otimes\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu)$. Furthermore, $\text{cov}\{\mathbf{c}_{\boldsymbol{\epsilon}}(l;\nu),\mathbf{c}_{\boldsymbol{\epsilon}}(m;\nu)\} = \mathbf{0}$, $l\neq m$.

By the application of a central limit theorem for martingale difference sequences (see Hamilton (1994, p. 194) or Hannan (1970, p. 228)), the vector $\mathbf{c}_{\boldsymbol{\epsilon}}(\nu)$ follows asymptotically a d^2M-variate normal distribution :

$$N^{1/2}\mathbf{c}_{\boldsymbol{\epsilon}}(\nu)\xrightarrow{d}N_{d^2M}(\mathbf{0},\mathbf{V}(\nu;M)\otimes\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu)), \tag{3.27}$$

where $\mathbf{V}(\nu; M)$ corresponds to the $(dM) \times (dM)$ block diagonal matrix :

$$\mathbf{V}(\nu; M) = \begin{pmatrix} \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu-1) & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu-2) & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & & & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu-M) \end{pmatrix}. \tag{3.28}$$

Interestingly, if $\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu) \equiv \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}$, the periodic white noise $\{\boldsymbol{\epsilon}_t\}$ reduces to a usual white noise composed of independent zero mean random vectors with the same covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}$. Thus, in this particular case, the asymptotic covariance matrix in the asymptotic distribution stated in relation (3.27) reduces to $\mathbf{I}_M \otimes \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}} \otimes \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}$, and we retrieve a theorem due to Li and McLeod (1981); our result generalizes their multivariate result to the periodic framework. See also Hipel and McLeod (1994, p. 499) for a discussion of a similar result for univariate periodic time series models.

From (3.25), we note that we have the following relation :

$$N^{1/2}\{\hat{\boldsymbol{\beta}}(\nu) - \boldsymbol{\beta}(\nu)\} = \{\boldsymbol{\Omega}^{-1}(\nu) \otimes \mathbf{I}_d\} N^{-1/2}\{\mathbf{X}(\nu) \otimes \mathbf{I}_d\}\mathbf{e}(\nu) + \mathbf{o}_P(1).$$

From this we deduce that :

$$\lim_{N\to\infty} N\mathrm{cov}\{\hat{\boldsymbol{\beta}}(\nu) - \boldsymbol{\beta}(\nu), \mathbf{c}_{\boldsymbol{\epsilon}}(\nu)\} = \{\boldsymbol{\Omega}^{-1}(\nu) \otimes \mathbf{I}_d\} \lim_{N\to\infty} E\left[\mathrm{vec}\{\mathbf{E}(\nu)\mathbf{X}^\top(\nu)\}\mathbf{c}_{\boldsymbol{\epsilon}}^\top(\nu)\right].$$

Vectorizing,

$$\begin{aligned} \mathrm{vec}\{\mathbf{E}(\nu)\mathbf{X}^\top(\nu)\} &= \sum_{n=0}^{N-1} \mathrm{vec}\{\boldsymbol{\epsilon}_{ns+\nu}\mathbf{X}_n^\top(\nu)\}, \\ &= \sum_{n=0}^{N-1} \{\mathbf{X}_n(\nu) \otimes \boldsymbol{\epsilon}_{ns+\nu}\}. \end{aligned} \tag{3.29}$$

Consequently, (3.26) and (3.29) give :

$$\lim_{N\to\infty} E\left[\mathrm{vec}\{\mathbf{E}(\nu)\mathbf{X}^\top(\nu)\}\mathbf{c}_{\boldsymbol{\epsilon}}^\top(l;\nu)\right] =$$
$$\lim_{N\to\infty} \left[N^{-1}\sum_{n=l}^{N-1} E\{\mathbf{X}_n(\nu)\boldsymbol{\epsilon}_{ns+\nu-l}^\top\}\right] \otimes \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu) = \mathbf{G}(l;\nu) \otimes \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu),$$

where $\mathbf{G}(l;\nu) = E\{\mathbf{X}_n(\nu)\boldsymbol{\epsilon}_{ns+\nu-l}^\top\}$.

Using the infinite moving average representation (3.5), we obtain that :

$$\begin{aligned} E(\mathbf{Y}_{ns+\nu-l'}\boldsymbol{\epsilon}_{ns+\nu-l}^{\top}) &= \sum_{k=0}^{\infty}\boldsymbol{\Psi}_k(\nu-l')E(\boldsymbol{\epsilon}_{ns+\nu-l'-k}\boldsymbol{\epsilon}_{ns+\nu-l}^{\top}), \\ &= \boldsymbol{\Psi}_{l-l'}(\nu-l')\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu-l), \end{aligned}$$

and $\mathbf{G}(l;\nu)$ represents a $\{dp(\nu)\} \times d$ matrix satisfying :

$$\mathbf{G}(l;\nu) = \begin{pmatrix} \boldsymbol{\Psi}_{l-1}(\nu-1) \\ \vdots \\ \boldsymbol{\Psi}_{l-p(\nu)}\{\nu-p(\nu)\} \end{pmatrix} \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu-l),$$

where $\boldsymbol{\Psi}_0(\nu) = \mathbf{I}_d$ and $\boldsymbol{\Psi}_k(\nu) = \mathbf{0}$ for $k < 0$, $\nu = 1, \ldots, s$. Collecting $\mathbf{G}(l;\nu)$, $l = 1, \ldots, M$, in a $\{dp(\nu)\} \times (Md)$ matrix leads to the expression :

$$\mathbf{G}(\nu) = (\mathbf{G}(1;\nu), \ldots, \mathbf{G}(M;\nu)). \tag{3.30}$$

Next, we state the joint asymptotic distribution of $N^{-1/2}\text{vec}\{\mathbf{E}(\nu)\mathbf{X}^{\top}(\nu)\}$ and $N^{1/2}\mathbf{c}_{\boldsymbol{\epsilon}}(\nu)$.

Proposition 3.1. *Let $\mathbf{E}(\nu)$ and $\mathbf{X}(\nu)$ be defined by (3.14) and (3.15), respectively. Consider a vector of M sample autocovariances collected in the vector $\mathbf{c}_{\boldsymbol{\epsilon}}(\nu)$ given by (3.12). Suppose that $\{\mathbf{Y}_t\}$ denotes a PVAR process satisfying (3.1), with a periodic white noise $\boldsymbol{\epsilon}$ as described in Theorem 3.1. Then :*

$$\begin{pmatrix} N^{-1/2}\text{vec}\{\mathbf{E}(\nu)\mathbf{X}^{\top}(\nu)\} \\ N^{1/2}\mathbf{c}_{\boldsymbol{\epsilon}}(\nu) \end{pmatrix} \xrightarrow{d} N_{d^2\{p(\nu)+M\}}\left(\mathbf{0}, \begin{pmatrix} \boldsymbol{\Omega}(\nu)\otimes\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu) & \mathbf{G}(\nu)\otimes\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu) \\ \mathbf{G}^{\top}(\nu)\otimes\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu) & \mathbf{V}(\nu;M)\otimes\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu) \end{pmatrix}\right),$$

where $\mathbf{V}(\nu;M)$ and $\mathbf{G}(\nu)$ correspond to the expressions (3.28) and (3.30), respectively. The matrix $\boldsymbol{\Omega}(\nu)$ is defined in Theorem 3.1.

The proof of Proposition 3.1 follows using the same kind of arguments as those found in Ahn (1988) and it is therefore omitted. See also Lütkepohl (2005, p. 165).

We now discuss the asymptotic distribution of $N^{1/2}\mathbf{c}_{\hat{\boldsymbol{\epsilon}}}(\nu)$ where there is no constraint on the parameters. By expanding $\mathbf{c}_{\hat{\boldsymbol{\epsilon}}}(\nu)$ in a Taylor expansion around $\hat{\boldsymbol{\beta}}(\nu)$ and evaluating at the point $\hat{\boldsymbol{\beta}}(\nu) = \hat{\boldsymbol{\beta}}(\nu)$, we obtain the following development :

$$\mathbf{c}_{\hat{\boldsymbol{\epsilon}}}(\nu) = \mathbf{c}_{\boldsymbol{\epsilon}}(\nu) + \frac{\partial \mathbf{c}_{\hat{\boldsymbol{\epsilon}}}(\nu)}{\partial \boldsymbol{\beta}^{\top}(\nu)}\left\{\hat{\boldsymbol{\beta}}(\nu) - \boldsymbol{\beta}(\nu)\right\} + \mathbf{o}_p(N^{-1/2}),$$

where $\partial \mathbf{c}_{\dot{\boldsymbol{\epsilon}}}(\nu)/\partial \boldsymbol{\beta}^\top(\nu)$ corresponds to a $(d^2M) \times \{d^2 p(\nu)\}$ matrix satisfying

$$\left(\partial \mathbf{c}_{\dot{\boldsymbol{\epsilon}}}(\nu)/\partial \boldsymbol{\beta}^\top(\nu)\right)^\top = \left((\partial \mathbf{c}_{\dot{\boldsymbol{\epsilon}}}(1;\nu)/\partial \boldsymbol{\beta}^\top(\nu))^\top, \ldots, (\partial \mathbf{c}_{\dot{\boldsymbol{\epsilon}}}(M;\nu)/\partial \boldsymbol{\beta}^\top(\nu))^\top\right).$$

Proceeding as in Duchesne (2005), an application of the law of large numbers for martingale difference sequences gives :

$$\frac{\partial \mathbf{c}_{\dot{\boldsymbol{\epsilon}}}(\nu)}{\partial \boldsymbol{\beta}^\top(\nu)} \xrightarrow{p} -\mathbf{G}^\top(\nu) \otimes \mathbf{I}_d.$$

Consequently, $N^{1/2}\mathbf{c}_{\dot{\boldsymbol{\epsilon}}}(\nu)$ and $N^{1/2}[\mathbf{c}_{\boldsymbol{\epsilon}}(\nu) - \{\mathbf{G}^\top(\nu) \otimes \mathbf{I}_d\}\{\hat{\boldsymbol{\beta}}(\nu) - \boldsymbol{\beta}(\nu)\}]$ have the same asymptotic distribution and, as a corollary of Proposition 3.1, we easily deduce that :

$$N^{1/2}\mathbf{c}_{\dot{\boldsymbol{\epsilon}}}(\nu) \xrightarrow{d} N_{d^2M}\left(\mathbf{0}, \boldsymbol{\Delta}_U(\nu)\right), \tag{3.31}$$

where

$$\boldsymbol{\Delta}_U(\nu) = \mathbf{V}(\nu; M) \otimes \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu) - \mathbf{G}^\top(\nu)\boldsymbol{\Omega}^{-1}(\nu)\mathbf{G}(\nu) \otimes \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu).$$

When the parameters satisfy the linear constraints (3.10), similar calculations give :

$$\frac{\partial \mathbf{c}_{\dot{\boldsymbol{\epsilon}}}(\nu)}{\partial \boldsymbol{\xi}^\top(\nu)} \xrightarrow{p} -\{\mathbf{G}^\top(\nu) \otimes \mathbf{I}_d\}\mathbf{R}(\nu),$$

and using the same kind of arguments we obtain the following asymptotic distribution :

$$N^{1/2}\mathbf{c}_{\dot{\boldsymbol{\epsilon}}}(\nu) \xrightarrow{d} N_{d^2M}\left(\mathbf{0}, \boldsymbol{\Delta}_R(\nu)\right), \tag{3.32}$$

where

$$\begin{aligned}\boldsymbol{\Delta}_R(\nu) \;=\; & \mathbf{V}(\nu; M) \otimes \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu) - \\ & \{\mathbf{G}^\top(\nu) \otimes \mathbf{I}_d\}\mathbf{R}(\nu)[\mathbf{R}^\top(\nu)\{\boldsymbol{\Omega}(\nu) \otimes \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\nu)\}\mathbf{R}(\nu)]^{-1}\mathbf{R}^\top(\nu)\{\mathbf{G}(\nu) \otimes \mathbf{I}_d\}.\end{aligned}$$

Let

$$\mathbf{L}(\nu) = \begin{pmatrix} \boldsymbol{\Gamma}_0^{-1}(\nu-1) & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Gamma}_0^{-1}(\nu-2) & \mathbf{0} & \ldots & \mathbf{0} \\ \vdots & & & \ddots & \vdots \\ \mathbf{0} & & & \ldots & \boldsymbol{\Gamma}_0^{-1}(\nu-M) \end{pmatrix} \otimes \boldsymbol{\Gamma}_0^{-1}(\nu).$$

Since $\mathbf{D}_{\dot{\boldsymbol{\epsilon}}}(\nu)$ converges to $\boldsymbol{\Gamma}_0(\nu)$ in probability, that is $\mathbf{D}_{\dot{\boldsymbol{\epsilon}}}(\nu) \xrightarrow{p} \boldsymbol{\Gamma}_0(\nu)$, the asymptotic distribution of $N^{1/2}\mathbf{r}_{\dot{\boldsymbol{\epsilon}}}(\nu)$ is obtained by an appropriate scaling of

results (3.31) and (3.32). More precisely, in the unconstrained and constrained case, the asymptotic distribution of the residual autocorrelation matrices are given by :

$$N^{1/2}\mathbf{r}_{\hat{\boldsymbol{\epsilon}}}(\nu) \xrightarrow{d} N_{d^2M}\left(\mathbf{0}, \mathbf{L}(\nu)\boldsymbol{\Delta}_U(\nu)\mathbf{L}^\top(\nu)\right),$$
$$N^{1/2}\mathbf{r}_{\hat{\boldsymbol{\epsilon}}}(\nu) \xrightarrow{d} N_{d^2M}\left(\mathbf{0}, \mathbf{L}(\nu)\boldsymbol{\Delta}_R(\nu)\mathbf{L}^\top(\nu)\right),$$

respectively.

These results on the asymptotic distribution of the residual autocovariance and autocorrelation matrices for PVAR time series models generalize the theorems of Li and McLeod (1981) in the context of VAR models. See also Lütkepohl (2005, Proposition 5.7) for similar results in VAR models when the model parameters satisfy linear constraints ; and see Duchesne (2005) for the asymptotic distribution of the residual autocovariance and autocorrelation matrices for VAR models with exogenous variables and linear constraints on the model parameters.

We now introduce the $d^2 \times d^2$ matrices $\mathbf{P}_i$, $i = 1, \ldots, M$, such that :

$$\begin{aligned} \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\nu - i) \otimes \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\nu) &= \mathbf{P}_i^\top \mathbf{P}_i, \; i = 1, \ldots, M, \\ \mathbf{P}_i\{\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu - i) \otimes \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu)\}\mathbf{P}_i^\top &= \mathbf{I}_{d^2}. \end{aligned}$$

Let the block-diagonal matrix $\mathbf{Q}_M$ be defined by :

$$\mathbf{Q}_M = \begin{pmatrix} \mathbf{P}_1 & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_2 & \mathbf{0} & \ldots & \mathbf{0} \\ \vdots & & & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{P}_M \end{pmatrix}, \tag{3.33}$$

and consider $\tilde{\mathbf{c}}_{\hat{\boldsymbol{\epsilon}}}(\nu) = \mathbf{Q}_M \mathbf{c}_{\hat{\boldsymbol{\epsilon}}}(\nu)$. It follows that the asymptotic covariance matrix of $\tilde{\mathbf{c}}_{\hat{\boldsymbol{\epsilon}}}(\nu)$ is given by :

$$\lim_{N\to\infty} N\mathrm{var}\{\tilde{\mathbf{c}}_{\hat{\boldsymbol{\epsilon}}}(\nu)\} = \mathbf{I}_{d^2M} - \mathbf{Q}_M\left\{\mathbf{G}^\top(\nu)\boldsymbol{\Omega}^{-1}(\nu)\mathbf{G}(\nu) \otimes \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu)\right\}\mathbf{Q}_M^\top.$$

Reworking the arguments of Li and McLeod (1981, pp. 235-236), the following relation holds approximately :

$$\boldsymbol{\Omega}(\nu) \approx \mathbf{G}(\nu)\mathbf{V}^{-1}(\nu; M)\mathbf{G}^\top(\nu), \tag{3.34}$$

where we used the relation $\mathbf{Q}_M^\top \mathbf{Q}_M = \mathbf{V}^{-1}(\nu; M) \otimes \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\nu)$; from this we deduce that the asymptotic covariance matrix of $\hat{\mathbf{c}}_{\hat{\boldsymbol{\epsilon}}}(\nu)$ is approximatively idempotent.

The asymptotic distributions (3.31) and (3.32) are useful to test the joint statistical significance of $\mathbf{c}_{\boldsymbol{\epsilon}}(l;\nu)$, $l = 1, \ldots, M$. More formally, the null hypothesis of model adequacy is given by :

$$H_0 : \boldsymbol{\gamma}_{\boldsymbol{\epsilon}}(\nu) = \mathbf{0},$$

where $\boldsymbol{\gamma}_{\boldsymbol{\epsilon}}(\nu)$ is defined by (3.11) and $\mathbf{0}$ corresponds to the $(d^2 M) \times 1$ null vector. Let $\hat{\mathbf{V}}(\nu; M)$ be a consistent estimator of $\mathbf{V}(\nu; M)$, obtained by estimating consistently $\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu - l)$ by $\hat{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon}}(\nu - l)$, $l = 1, \ldots, M$. A portmanteau test statistic relying on $\mathbf{c}_{\hat{\boldsymbol{\epsilon}}}(\nu)$ is given by :

$$\begin{aligned} Q_M(\nu) &= N \mathbf{c}_{\hat{\boldsymbol{\epsilon}}}^\top(\nu) \left\{ \hat{\mathbf{V}}^{-1}(\nu; M) \otimes \hat{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon}}^{-1}(\nu) \right\} \mathbf{c}_{\hat{\boldsymbol{\epsilon}}}(\nu), \qquad (3.35) \\ &= N \sum_{l=1}^{M} \mathbf{c}_{\hat{\boldsymbol{\epsilon}}}^\top(l;\nu) \left\{ \hat{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon}}^{-1}(\nu - l) \otimes \hat{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon}}^{-1}(\nu) \right\} \mathbf{c}_{\hat{\boldsymbol{\epsilon}}}(l;\nu), \\ &= N \sum_{l=1}^{M} \mathrm{tr}\{ \mathbf{C}_{\hat{\boldsymbol{\epsilon}}}^\top(l;\nu) \hat{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon}}^{-1}(\nu) \mathbf{C}_{\hat{\boldsymbol{\epsilon}}}(l;\nu) \hat{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon}}^{-1}(\nu - l) \}. \end{aligned}$$

The last equality is obtained from the following result on matrix calculus (Harville (1997, Theorem 16.2.2)) :

$$\mathrm{tr}(\mathbf{A}^\top \mathbf{B} \mathbf{C} \mathbf{D}^\top) = \{\mathrm{vec}(\mathbf{A})\}^\top \{\mathbf{D} \otimes \mathbf{B}\} \{\mathrm{vec}(\mathbf{C})\},$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are any matrices for which the above product is defined. The test statistic $Q_M(\nu)$ follows approximatively a chi-square distribution $\chi^2_{d^2\{M - p(\nu)\}}$, where χ^2_d denotes a chi-square distribution with d degrees of freedom.

As discussed in McLeod (1994) in the univariate case, the Ljung-Box correction factor $N/\{N - \lfloor (l - \nu + s)/s \rfloor\}$ is expected to improve the finite-sample properties of the test statistic (3.35), where $\lfloor x \rfloor$ represents the integer part of the real number x. This leads to the finite-sample corrected test statistic :

$$Q_M^*(\nu) = N \sum_{l=1}^{M} \frac{N}{N - \lfloor (l - \nu + s)/s \rfloor} \mathrm{tr}\{ \mathbf{C}_{\hat{\boldsymbol{\epsilon}}}^\top(l;\nu) \hat{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon}}^{-1}(\nu) \mathbf{C}_{\hat{\boldsymbol{\epsilon}}}(l;\nu) \hat{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon}}^{-1}(\nu - l) \}. \qquad (3.36)$$

When the parameters are supposed to satisfy the linear constraints given by (3.10), a direct corollary of (3.34) is that the asymptotic covariance matrix of $N^{1/2}\tilde{\mathbf{c}}_{\hat{\boldsymbol{\epsilon}}}$ is approximatively idempotent of rank $d^2M - K(\nu)$. Consequently, the asymptotic distribution of the test statistics $Q_M(\nu)$ and $Q_M^*(\nu)$ defined by (3.35) and (3.36) under the linear constraints (3.10) is approximatively the chi-square distribution $\chi^2_{d^2M-K(\nu)}$.

These portmanteau test statistics designed to check PVAR time series models generalize test statistics originally proposed by Hosking (1980) and Li and McLeod (1981) in the context of VAR models. See also Li (2004, Chapter 3). Another relevant result is established in Lütkepohl (2005, Proposition 5.8), who states the approximate distribution of portmanteau test statistics for VAR models, when the model parameters satisfy linear constraints; our results generalize that work for periodic time series.

The test statistics (3.35) and (3.36) are asymptotically independent across the seasons $\nu = 1, 2, \ldots, s$. Consequently, global test statistics which can be used to test the null hypothesis of model adequacy for all seasons taken simultaneously can be constructed by summing (3.35) or (3.36) over all seasons :

$$Q_M = \sum_{\nu=1}^{s} Q_M(\nu), \tag{3.37}$$

$$Q_M^* = \sum_{\nu=1}^{s} Q_M^*(\nu). \tag{3.38}$$

In the unrestricted case, the test statistics Q_M and Q_M^* follow approximatively a chi-square distribution with $d^2 \sum_{\nu=1}^{s}\{M - p(\nu)\}$ degrees of freedom. When linear constraints of the form (3.10) are on the parameters, the chi-square approximation is still valid but the degrees of freedom are $\sum_{\nu=1}^{s}\{d^2M - K(\nu)\}$. These global test statistics represent natural extensions of proposals suggested originally by Hipel and McLeod (1994, p. 500) in the univariate case. In the next section, the test statistics proposed in this section are illustrated in a small empirical study.

3.5. Simulation experiments

In the previous section, we presented portmanteau test statistics which should prove useful in diagnosing PVAR models. It is natural to provide empirical evidence by evaluating the finite-sample properties of the proposed test statistics. Here, we report the simulation results of a small Monte Carlo experiment conducted in order to study their exact levels. We included in our study the test statistics calculated at each season, that is $Q_M(\nu)$ and $Q_M^*(\nu)$, $\nu = 1, \ldots, s$, and also the global versions Q_M and Q_M^*. To compare the exact distribution of the test statistics with their corresponding χ^2 distributions, the following bivariate data generating processes (DGP) were used :

$$\text{DGP}_1 : \mathbf{Y}_{ns+\nu} = \mathbf{\Phi}(\nu)\mathbf{Y}_{ns+\nu-1} + \boldsymbol{\epsilon}_{ns+\nu}, \tag{3.39}$$

$$\text{DGP}_2 : \mathbf{Y}_{ns+\nu} = \mathbf{\Phi}_C(\nu)\mathbf{Y}_{ns+\nu-1} + \boldsymbol{\epsilon}_{ns+\nu}. \tag{3.40}$$

We considered the case of four seasons, that is $\nu = 4$. The autoregressive coefficients of models DGP_1 and DGP_2 are given by :

$$\mathbf{\Phi}(1) = \begin{pmatrix} 0.30 & 0.10 \\ 0.10 & 0.20 \end{pmatrix}, \ \mathbf{\Phi}(2) = \begin{pmatrix} 0.42 & 0.24 \\ -0.20 & 0.50 \end{pmatrix},$$

$$\mathbf{\Phi}(3) = \begin{pmatrix} -0.80 & 0.20 \\ 0.60 & 0.70 \end{pmatrix}, \ \mathbf{\Phi}(4) = \begin{pmatrix} -0.30 & 0.50 \\ 0.90 & -0.20 \end{pmatrix},$$

$$\mathbf{\Phi}_C(1) = \begin{pmatrix} 0.30 & 0.00 \\ 1.00 & 0.20 \end{pmatrix}, \ \mathbf{\Phi}_C(2) = \begin{pmatrix} 0.42 & 0.00 \\ -0.20 & 0.50 \end{pmatrix},$$

$$\mathbf{\Phi}_C(3) = \begin{pmatrix} -0.80 & 0.00 \\ 0.60 & 0.70 \end{pmatrix}, \ \mathbf{\Phi}_C(4) = \begin{pmatrix} -0.30 & 0.00 \\ 0.90 & -0.20 \end{pmatrix}.$$

The process $\boldsymbol{\epsilon} = \{\boldsymbol{\epsilon}_t, t \in \mathbb{Z}\}$ was assumed to be a periodic Gaussian white noise, composed of independent Gaussian random vectors with mean $\mathbf{0}$ and covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu)$, where the covariance matrices satisfy :

$$\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(1) = \begin{pmatrix} 1.00 & 0.50 \\ 0.50 & 1.50 \end{pmatrix}, \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(2) = \begin{pmatrix} 1.60 & 0.30 \\ 0.30 & 0.50 \end{pmatrix},$$

$$\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(3) = \begin{pmatrix} 0.20 & 0.10 \\ 0.10 & 0.80 \end{pmatrix}, \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(4) = \begin{pmatrix} 0.50 & 0.10 \\ 0.10 & 0.20 \end{pmatrix}.$$

For the purposes of our illustration, no parameter constraints were hypothesized for DGP_1, but for DGP_2 it was assumed that the zero-valued parameters in $\boldsymbol{\Phi}_C(1)$, $\boldsymbol{\Phi}_C(2)$, $\boldsymbol{\Phi}_C(3)$ and $\boldsymbol{\Phi}_C(4)$ were known.

We examined the empirical frequencies of rejection of the null hypothesis of adequacy at two different nominal levels (5 and 10 percent) for each of two series lengths ($N = 200$ and $N = 400$ observations by season). For each series length, 10000 independent realizations were generated.

For each realization of the DGP defined by (3.39), a PVAR model of order one was estimated by least squares estimators, as described in Section 3.3.1. When the DGP was given by (3.40), the zero-valued parameters in $\boldsymbol{\Phi}_C(\nu)$ were taken into account by properly defining the constraint matrix $\mathbf{R}(\nu)$, $\nu = 1, 2, 3, 4$, and the parameters were estimated using the two-step procedure described in Section 3.3.2.

For each residual time series, the portmanteau test statistics $Q_M(\nu)$ and $Q_M^*(\nu)$ and the global portmanteau test statistics Q_M and Q_M^* were calculated for $M = 5, 10, 15, 20, 25, 30, 35, 40, 45, 50$. For each nominal level and for each series of length $n = 4N$, we obtained from the 10000 realizations the empirical frequencies of rejection of the null hypothesis of adequacy. The standard errors of the empirical levels based on 10000 independent realizations are 0.218% and 0.300% for the nominal levels 5% and 10%, respectively.

The empirical levels of the portmanteau test statistics $Q_M(\nu)$ and $Q_M^*(\nu)$ for the PVAR models without and with parameter constraints are presented in Tables 1 and 2, respectively. As expected, the test statistics $Q_M^*(\nu)$ exhibited better empirical levels than $Q_M(\nu)$: as in the PAR case, the factor correction

TABLE 3.1. Empirical levels (in percentage) of the portmanteau test statistics $Q_M(\nu)$ defined by (3.35), and its modified version $Q^*_M(\nu)$ defined by (3.36), for the PVAR model without constraints on the parameters, given by (3.39).

	$\alpha = 0.05$															
	$N = 200$								$N = 400$							
	$Q_M(\nu)$				$Q^*_M(\nu)$				$Q_M(\nu)$				$Q^*_M(\nu)$			
M \ ν	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
5	4.81	4.84	5.24	5.32	5.12	5.11	5.48	5.43	4.73	5.19	5.06	4.96	4.86	5.28	5.15	5.06
10	4.65	4.69	5.00	4.99	5.22	5.23	5.40	5.41	4.33	4.85	4.59	4.74	4.67	5.13	4.84	4.92
15	4.40	4.75	4.49	4.79	5.29	5.62	5.07	5.54	4.13	4.95	4.70	4.61	4.53	5.34	4.96	4.88
20	4.05	4.15	4.37	4.33	5.23	5.23	5.43	5.39	4.18	4.64	4.38	4.26	4.77	5.34	4.85	4.78
25	3.74	3.70	4.11	3.98	5.19	5.14	5.45	5.17	4.38	4.40	4.32	4.05	5.07	5.13	4.95	4.78
30	3.68	3.28	3.86	3.61	5.52	5.10	5.76	5.26	4.00	4.18	4.17	4.21	4.93	4.99	4.90	4.92
35	3.15	3.26	3.63	3.34	5.49	5.43	5.68	5.37	3.95	4.16	3.83	3.81	4.99	5.18	4.85	4.85
40	2.88	3.15	3.27	3.12	5.36	5.63	5.79	5.41	3.77	3.92	3.86	3.80	5.27	5.18	5.11	4.91
45	2.54	2.63	3.06	2.69	5.41	5.78	5.93	5.32	3.69	3.80	3.47	3.91	5.15	5.28	5.15	5.06
50	2.32	2.38	2.91	2.33	5.48	5.38	6.24	5.41	3.66	3.49	3.37	3.63	5.23	5.28	4.98	5.22
	$\alpha = 0.10$															
	$N = 200$								$N = 400$							
	$Q_M(\nu)$				$Q^*_M(\nu)$				$Q_M(\nu)$				$Q^*_M(\nu)$			
M \ ν	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
5	10.14	9.46	10.22	10.40	10.62	9.85	10.65	10.77	9.81	10.15	9.67	10.51	10.07	10.42	9.86	10.65
10	9.58	9.60	9.66	9.93	10.57	10.57	10.45	10.63	9.24	9.61	9.48	9.76	9.76	10.07	9.84	10.17
15	9.39	9.59	8.83	9.45	10.70	10.94	10.09	10.56	8.73	9.55	9.27	9.35	9.37	10.12	9.92	9.92
20	8.45	8.81	8.76	8.54	10.62	10.39	10.48	10.23	8.60	9.12	9.01	9.02	9.73	10.01	9.81	9.85
25	7.85	7.97	8.39	8.35	10.50	10.58	10.71	10.56	8.86	9.02	9.08	8.76	10.31	10.32	10.30	9.82
30	7.57	7.30	8.11	7.76	10.73	10.59	11.13	10.43	8.22	8.53	8.53	8.59	9.83	10.33	10.02	9.88
35	7.07	7.04	7.54	7.46	10.62	10.41	11.05	10.80	8.03	8.25	7.92	8.43	9.89	10.08	10.07	10.07
40	6.26	6.69	6.91	6.75	10.53	10.73	11.39	11.10	7.94	7.85	7.96	8.09	10.04	10.24	10.29	10.10
45	5.71	6.06	6.56	6.22	10.77	10.73	11.85	10.81	7.61	7.80	7.38	7.60	10.35	10.37	9.90	9.84
50	5.24	5.28	6.30	5.65	10.65	10.72	12.01	10.66	7.47	7.50	7.36	7.38	10.61	10.63	10.48	9.88

proposed by McLeod (1994) improved the χ^2 approximation for the test statistic $Q^*_M(\nu)$, offering generally better finite sample properties than $Q_M(\nu)$, particularly as M increases. We concentrate the rest of our discussion on $Q^*_M(\nu)$ only. Generally, the χ^2 distribution provided a satisfactory approximation for all lags,

TABLE 3.2. Empirical levels (in percentage) of the portmanteau test statistics $Q_M(\nu)$ defined by (3.35), and its modified version $Q_M^*(\nu)$ defined by (3.36), for the PVAR model with constraints on the parameters, given by (3.40).

	$\alpha = 0.05$															
	$N = 200$								$N = 400$							
	$Q_M(\nu)$				$Q_M^*(\nu)$				$Q_M(\nu)$				$Q_M^*(\nu)$			
M \ ν	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
5	5.11	4.43	5.21	5.30	5.36	4.70	5.45	5.55	4.71	5.63	4.92	4.93	4.86	5.74	5.02	5.03
10	4.57	4.67	4.77	5.01	5.17	5.16	5.27	5.36	4.39	4.99	4.51	4.72	4.72	5.25	4.70	4.96
15	4.41	4.60	4.41	4.92	5.38	5.45	5.05	5.59	4.21	4.94	4.72	4.73	4.66	5.33	5.01	4.97
20	3.92	4.15	4.37	4.36	5.22	5.23	5.39	5.29	4.21	4.81	4.40	4.37	4.72	5.25	4.86	4.87
25	3.86	3.78	4.00	3.86	5.10	5.05	5.43	5.22	4.34	4.43	4.30	4.10	5.11	5.22	4.90	4.83
30	3.59	3.36	3.70	3.65	5.55	5.24	5.64	5.07	3.91	4.20	4.12	4.29	4.99	5.08	4.87	5.06
35	3.05	3.23	3.72	3.25	5.70	5.32	5.73	5.36	3.91	4.10	3.99	3.88	4.99	5.17	4.94	4.89
40	2.89	3.00	3.29	3.13	5.65	5.37	5.77	5.49	3.81	3.86	3.90	3.80	5.01	5.16	5.13	4.90
45	2.65	2.62	3.23	2.63	5.70	5.70	5.99	5.41	3.73	3.81	3.41	3.85	5.29	5.19	4.89	5.03
50	2.29	2.32	3.00	2.38	5.52	5.35	6.09	5.53	3.57	3.62	3.50	3.66	5.34	5.24	4.93	5.16
	$\alpha = 0.10$															
	$N = 200$								$N = 400$							
	$Q_M(\nu)$				$Q_M^*(\nu)$				$Q_M(\nu)$				$Q_M^*(\nu)$			
M \ ν	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
5	9.89	9.32	10.15	10.28	10.50	9.75	10.49	10.58	9.45	10.46	10.01	10.20	9.80	10.60	10.14	10.29
10	9.77	9.46	9.59	10.23	10.81	10.23	10.31	10.81	9.24	9.71	9.67	9.69	9.76	10.01	9.96	9.95
15	9.48	9.40	9.13	9.47	10.80	10.78	10.31	10.65	8.56	9.67	9.12	9.61	9.36	10.31	9.72	10.15
20	8.72	8.48	8.78	8.63	10.68	10.45	10.49	10.23	8.91	9.34	8.97	9.04	9.82	10.24	9.91	9.96
25	8.05	7.67	8.41	8.31	10.61	10.30	10.77	10.53	8.97	9.06	9.08	8.99	10.12	10.30	10.25	9.85
30	7.57	7.41	8.20	7.68	10.82	10.15	11.18	10.32	8.32	8.54	8.55	8.69	9.91	10.11	10.11	10.19
35	7.21	6.98	7.65	7.48	10.86	10.22	11.21	10.83	8.01	8.28	8.06	8.40	10.05	9.95	10.02	10.19
40	6.43	6.58	7.12	6.89	10.61	10.59	11.55	10.92	8.00	7.82	7.90	8.02	9.83	10.16	10.11	10.05
45	5.89	6.06	6.70	6.30	10.72	10.57	11.76	10.99	7.62	7.54	7.43	7.72	10.17	10.20	9.92	9.95
50	5.19	5.15	6.17	5.78	10.70	10.75	11.91	10.98	7.43	7.46	7.40	7.54	10.37	10.32	10.51	10.01

at both significance levels. The results for the models without and with parameter constraints were very comparable. For $N = 200$, some overrejection has been observed at season $\nu = 3$ for DGP_1 and DGP_2, but in general the rejection rates at the 5% and 10% nominal levels are within two standard errors of 5% and

TABLE 3.3. Empirical levels (in percentage) of the global portmanteau test Q_M and Q_M^* defined by (3.37) and (3.38) for the PVAR model without constraints on the parameters, given by (3.39).

	$\alpha = 0.05$				$\alpha = 0.10$			
	$N = 200$		$N = 400$		$N = 200$		$N = 400$	
M	Q_M	Q_M^*	Q_M	Q_M^*	Q_M	Q_M^*	Q_M	Q_M^*
5	5.06	5.57	4.63	4.94	10.21	11.09	9.58	9.80
10	4.38	5.41	4.49	4.87	9.42	10.97	9.26	9.98
15	4.32	5.84	4.38	5.06	8.84	11.43	8.98	10.16
20	3.91	6.15	4.04	5.15	8.02	11.57	8.35	10.04
25	3.08	6.12	4.12	5.45	7.27	11.90	8.18	10.67
30	2.63	5.82	3.83	5.42	5.92	11.70	8.02	10.73
35	2.38	5.92	3.25	5.30	5.10	11.82	7.06	10.35
40	1.96	6.06	2.93	5.03	4.29	12.24	6.32	10.32
45	1.54	6.44	2.62	5.33	3.72	12.35	5.96	10.72
50	1.26	6.36	2.42	5.47	2.99	12.31	5.59	10.86

10%, respectively, or very close to these intervals. For $N = 400$, almost all corresponding empirical levels of $Q_M^*(\nu)$, $\nu = 1, 2, 3, 4$, lie within the 5% significant limits.

The empirical levels of the global portmanteau tests for PVAR models without and with parameter constraints are presented in Tables 3 and 4, respectively. The test statistic Q_M^* displayed better empirical levels than the uncorrected test statistic Q_M. As the test statistics calculated at each season, it appeared that some overrejection has been observed for the corrected test statistic for large M when $N = 200$. However, the empirical performance of (3.38) appeared more satisfactory than the uncorrected version (3.37). It seems that moderate to large sample sizes are needed in order to have a satisfactory performance for the global test statistics : for $N = 400$, almost all corresponding empirical levels of Q_M^* lie within the 5% significant limits.

TABLE 3.4. Empirical levels (in percentage) of the global portmanteau test Q_M and Q_M^* defined by (3.37) and (3.38) for the PVAR model with constraints on the parameters, given by (3.40).

	$\alpha = 0.05$				$\alpha = 0.10$			
	$N = 200$		$N = 400$		$N = 200$		$N = 400$	
M	Q_M	Q_M^*	Q_M	Q_M^*	Q_M	Q_M^*	Q_M	Q_M^*
5	5.13	5.59	4.74	4.84	10.23	11.00	9.63	9.86
10	4.76	5.69	4.58	5.02	9.53	11.04	9.47	10.08
15	4.48	5.83	4.40	5.11	9.01	11.37	9.12	10.33
20	4.09	6.01	4.01	5.06	7.97	11.40	8.48	10.27
25	3.39	6.12	4.21	5.49	7.25	11.75	8.24	10.59
30	2.74	6.09	3.82	5.53	6.19	11.52	7.90	10.73
35	2.40	6.32	3.37	5.32	5.43	11.92	6.99	10.50
40	1.95	6.40	2.91	5.26	4.50	12.19	6.29	10.31
45	1.68	6.43	2.70	5.28	3.81	12.27	6.04	10.81
50	1.30	6.45	2.40	5.45	3.00	12.19	5.52	10.82

From this limited empirical study, the finite sample performance of the test statistics seem rather reasonable, particularly for moderate to large sample sizes. Given the number of parameters involved in vector periodic time series, it is not really surprising that moderate to large sample sizes are needed in order to have satisfactory results. Overall, $Q_M^*(\nu)$, $\nu = 1, \ldots, s$ and Q_M^* can be recommended for diagnosing PVAR models.

3.6. APPLICATION WITH THE WEST GERMAN DATA

We illustrate here the new portmanteau test statistics with a real data set derived from the economic literature. The two variables are the quarterly seasonally unadjusted West German income and consumption data for the years 1960-1987. These data were analyzed by Lütkepohl (2005, Chapter 17.3.3) who aimed at testing the null hypothesis of constant coefficients, that is if a classical VAR model seemed appropriate for the West German data.

In order to have stationarity, the data are transformed by applying the first difference of the logarithm for each variable, giving a bivariate time series of sample size equal to $n = 111$. Because we analyze quarterly data, the period $\nu = 4$ is naturally selected. The two time series are displayed in Lütkepohl (2005, Figure 17.1). A strong seasonal pattern is observed.

Using several LR-test statistics, Lütkepohl (2005) rejected the null hypothesis of constant coefficients. From his results, he concludes that the testing methodology supported a periodic model. However, no attempt has been made to provide a periodic model for the data. The main goal here is to complement the analysis of Lütkepohl (2005), in fitting and checking the adequacy of PVAR models for the West German data, and to try to suggest a reasonable model for these data.

TABLE 3.5. Least squares estimators used to fit the West German data to a bivariate PVAR model with $\nu = 4$, such that the autoregressive orders, obtained by the BIC criterion defined by (3.41), are given by $(p(1), p(2), p(3), p(4)) = (2, 1, 3, 1)$, with constraints $\Phi_{1,11}(1) = 0$, $\Phi_{2,21}(1) = 0$, $\Phi_{2,12}(1) = 0$, $\Phi_{2,22}(1) = 0$, $\Phi_{1,21}(2) = 0$ and $\Phi_{1,22}(2) = 0$; the standard errors are given in parentheses.

$$\hat{\mathbf{\Phi}}_1(1) = \begin{pmatrix} 0 & -0.2955 \\ & (0.0499) \\ 0.3252 & -0.6959 \\ (0.1306) & (0.1066) \end{pmatrix}, \hat{\mathbf{\Phi}}_2(1) = \begin{pmatrix} -0.2812 & 0 \\ (0.0563) & \\ 0 & 0 \end{pmatrix},$$

$$\hat{\mathbf{\Phi}}_1(2) = \begin{pmatrix} 0.2247 & 0.1961 \\ (0.1499) & (0.0486) \\ 0 & 0 \end{pmatrix},$$

$$\hat{\mathbf{\Phi}}_1(3) = \begin{pmatrix} -1.1186 & 1.5694 \\ (0.3178) & (0.4515) \\ -0.4088 & 1.2369 \\ (0.3215) & (0.4568) \end{pmatrix}, \hat{\mathbf{\Phi}}_2(3) = \begin{pmatrix} -1.7824 & 2.0949 \\ (0.3680) & (0.3125) \\ -0.9379 & 1.5613 \\ (0.3723) & (0.3162) \end{pmatrix}, \hat{\mathbf{\Phi}}_3(3) = \begin{pmatrix} -1.1139 & 0.7885 \\ (0.1976) & (0.1923) \\ -0.4302 & 0.2505 \\ (0.1999) & (0.1945) \end{pmatrix},$$

$$\hat{\mathbf{\Phi}}_1(4) = \begin{pmatrix} -0.4055 & -0.6175 \\ (0.1268) & (0.1451) \\ 0.2686 & -1.6554 \\ (0.1213) & (0.1387) \end{pmatrix}.$$

TABLE 3.6. P-values of the portmanteau test statistics defined by (3.35) and (3.36) used to check a bivariate PVAR model with $\nu = 4$ for the West German data, such that the autoregressive orders, obtained by the BIC criterion defined by (3.41), are given by $(p(1), p(2), p(3), p(4)) = (2, 1, 3, 1)$, with constraints on the autoregressive parameters $\Phi_{1,11}(1) = 0$, $\Phi_{2,21}(1) = 0$, $\Phi_{2,12}(1) = 0$, $\Phi_{2,22}(1) = 0$, $\Phi_{1,21}(2) = 0$ and $\Phi_{1,22}(2) = 0$.

	$Q_M(\nu)$				$Q_M^*(\nu)$			
M \ ν	1	2	3	4	1	2	3	4
6	0.4598	0.1417	0.5787	0.1524	0.3969	0.1098	0.5310	0.1394
8	0.7236	0.2274	0.8110	0.1415	0.6499	0.1633	0.7532	0.1108
10	0.8650	0.4480	0.9265	0.1657	0.8001	0.3453	0.8887	0.1205
12	0.9323	0.6590	0.9819	0.2509	0.8801	0.5362	0.9646	0.1724
15	0.9823	0.8340	0.9769	0.4057	0.9550	0.7163	0.9429	0.2791
18	0.9925	0.8238	0.9528	0.3704	0.9694	0.6381	0.8533	0.1990
20	0.9944	0.8230	0.9647	0.4691	0.9679	0.5882	0.8544	0.2536

In view of the previous analysis, PVAR time series models with $\nu = 4$ are considered. The autoregressive orders $p(\nu)$, $\nu = 1, 2, 3, 4$ have been obtained by using the BIC information criterion (Akaike (1977), Schwarz (1978)). Generalizing to the multiplicative case ideas put forward by McLeod (1994), the BIC criterion may be factored to obtain a separate criterion for each period. More precisely, we define the BIC criterion as :

$$BIC = \sum_{\nu=1}^{s} BIC(\nu), \tag{3.41}$$

with

$$BIC(\nu) = \log\det \hat{\Sigma}_{\boldsymbol{\epsilon}}(\nu) + \frac{\log(N)}{N}\lambda(\nu),$$

where $\hat{\boldsymbol{\epsilon}}_{ns+\nu}$, $n = 0, \ldots, N-1$ denote the residuals of the adjustment, $\hat{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon}}(\nu)$ corresponds to the least squares estimators of $\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu)$, and $\lambda(\nu)$ represents the number of autoregressive parameters in the season ν. In the unconstrained model,

$\lambda(\nu) = d^2 p(\nu)$. In fitting an unconstrained model, $\hat{p}(\nu)$ gives an estimator of the autoregressive order $p(\nu)$ at season ν, and it is chosen such that the criterion is minimized, $\nu = 1, \ldots, s$. For the West German data, we found the autoregressive orders $(p(1), p(2), p(3), p(4)) = (2, 1, 3, 1)$.

Several models have been fitted using the two-step procedure presented in Section 3.3.2. First, a full unconstrained model was estimated with least squares estimators. A residual analysis was made and the portmanteau test statistics $Q_M(\nu)$ and $Q_M^*(\nu)$ were calculated. Generally, the model was satisfactory, relying on 28 independent parameters. The standard errors of the coefficients were determined, and several coefficients were not statistically significant from zero. In order to propose a more parsimonious model, each autoregressive parameter whose absolute value of the t-statistic (calculated as the value of the estimator divided by its standard error) was smaller than one was set to zero. Then, in a second step, the reduced PVAR model was estimated with the following constraints on the autoregressive parameters : $\Phi_{1,11}(1) = 0$, $\Phi_{2,21}(1) = 0$, $\Phi_{2,12}(1) = 0$, $\Phi_{2,22}(1) = 0$, $\Phi_{1,21}(2) = 0$ and $\Phi_{1,22}(2) = 0$. The least squares estimators of the final model are presented in Table 3.5. The residual analysis was done and the portmanteau test statistics $Q_M(\nu)$ and $Q_M^*(\nu)$ were calculated. The P-values are reported in Table 3.6. From Table 3.6, all the P-values suggest that the model was not rejected for the usual significance levels.

Note that Lütkepohl (2005), using selection criteria such as AIC and BIC, selected a VAR(p) model with $p = 5$ to carry out his testing methodology, where p was restricted to $p \in \{1, \ldots, 8\}$. This suggests that the best VAR model for these data is obtained by setting $p = 5$. We made a residual analysis and calculated the portmanteau test statistics of Hosking (1980) and Li and McLeod (1981), without and with the Ljung-Box adjustment. These test statistics are described in Lütkepohl (2005). The P-values of the test statistics for the adjustment of a VAR(5) model gave P-values smaller than any reasonable significance level, suggesting clearly an inappropriate model for these data. Interestingly, our final PVAR model relies on 22 independent parameters, that is only two more than the 20 parameters implied by the (inadequate) VAR(5) model.

Overall, these adjustments suggest that a PVAR model with linear parameter constraints seems reasonable for the West German data. Incidently, our analysis supports the findings of Lütkepohl (2005), who rejected the VAR model. However, for practical purposes, once the test statistics investigated by Lütkepohl (2005) have rejected the null hypothesis of a particular VAR model, it may be useful to provide explicitly a PVAR model, which could be used for forecasting purposes. Our methodology gives tools for a more complete description of the West German data and represents a useful complement to the approach described in Lütkepohl (2005).

3.7. Conclusion

In this paper, we studied least squares estimators in PVAR models with linear parameter constraints on a given season. Furthermore, we derived the asymptotic distribution of the residual autocovariance matrices in the class of PVAR models and obtained the asymptotic distribution of residual autocorrelation matrices. As applications of our asymptotic results, new test statistics of the portmanteau type have been presented. We introduced test statistics for each season and global versions as well. The proposed test statistics were illustrated in a small simulation study. From our simulation experiments, the test statistics with the Ljung-Box adjustment performed reasonably well and can be recommended for use. We applied our methodology using real data : the bivariate quarterly West German data. From the analysis of Lütkepohl (2005), the classical VAR model was clearly rejected but no periodic model was estimated. Using our approach, a reasonable model for modeling the quarterly West German data has been proposed and checked, using the portmanteau test statistics described in this paper. It is hoped that the results presented in this paper will be useful in practical applications, for fitting and diagnosing vector periodic time series models.

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Chapitre 4

ESTIMATION AND MODEL ADEQUACY CHECKING FOR MULTIVARIATE SEASONAL AUTOREGRESSIVE TIME SERIES MODELS WITH PERIODICALLY VARYING PARAMETERS

Cet article a été soumis pour publication en décembre 2007 et accepté en décembre 2008 dans la revue *Statistica Neerlandica*. Le premier auteur est Eugen Ursu et le coauteur est le directeur de recherche Pierre Duchesne.

Abstract : We introduce a class of multivariate seasonal time series models with periodically varying parameters, abbreviated by the acronym SPVAR. The model is suitable for multivariate data, and combines a periodic autoregressive structure and a multiplicative seasonal time series model. The stationarity conditions (in the periodic sense) and the theoretical autocovariance functions of SPVAR stochastic processes are derived. Estimation and checking stages are considered. The asymptotic normal distribution of the least squares estimators of the model parameters is established, and the asymptotic distributions of the residual autocovariance and autocorrelation matrices in the class of SPVAR time series models are obtained. In order to check model adequacy, portmanteau test statistics are considered and their asymptotic distributions are studied. A simulation study is

briefly discussed to investigate the finite-sample properties of the proposed test statistics. The methodology is illustrated with a bivariate quarterly data set on travellers entering to Canada.

Key words and phrases : Diagnostic checking; periodic time series; portmanteau test statistics; residual autocorrelation and autocovariance matrices; seasonal time series; vector time series.

Mathematics subject classification codes (2000) : primary 62M10; secondary 62H10.

4.1. INTRODUCTION

Time series with seasonal or periodic properties naturally arise in many fields, such as climatology, economics or hydrology, amongst others. In view of this, seasonal univariate time series, such as the seasonal autoregressive moving average (SARMA) time series model developed originally by Box and Jenkins (1970, Chap. 9), have been extensively studied in the literature. However, as noted by Reinsel (1997, p. 219), there has been much less investigation of seasonal modeling designed for multivariate data, by comparison to the univariate case. The multiplicative seasonal vector autoregressive moving average (SVARMA) models introduced by Reinsel (1997) represent a class of stationary models with large autocorrelation matrices (in norm) at lags which are multiple of the period s. However, a consequence of the stationarity property is that the autocorrelation matrices are invariant with respect to the season : lag h autocorrelations do not vary as a function of the season ν, $\nu = 1, \ldots, s$. On the other hand, many seasonal time series cannot be filtered to achieve second-order stationarity (see, e.g., Vecchia (1985a, 1985b)), the reason being that the correlation structure of these time series depends on the season. This is illustrated in McLeod (1993) for seasonal river flow time series : river flows for a particular season of the year may be statistically similar from year to year, but may depend intrinsically on the season, resulting in a periodic correlation structure. See also McLeod, Noakes, Hipel and Thompstone (1987).

Another class of models, periodic time series models, has received much attention in recent years. Periodic time series models are designed to handle time series data with periodic statistical structures. For univariate time series, pioneer work in the statistical literature has been realized by Jones and Brelsford (1967), Pagano (1978), Troutman (1979), Vecchia (1985a, 1985b), Vecchia and Ballerini (1991) and McLeod (1993, 1994), amongst others. More recently, Lund and Basawa (2000) explored recursive prediction and likelihood evaluation techniques for periodic autoregressive moving average (PARMA) models and Basawa and Lund (2001) studied large sample properties of the model parameters. The introduction of periodic models in the economic literature dates back to Parzen and Pagano

(1979), Osborn (1988) and Osborn and Smith (1989), amongst others. Many macroeconomic time series data display a trend ; periodic models for trending data and the relevant testing methodology have been developed by Boswijk and Franses (1996) and Paap and Franses (1999), and an extensive treatment is provided in the monograph of Franses and Paap (2004). For multivariate time series, Ula (1990, 1993) studied periodic covariance stationarity conditions for multivariate PARMA processes and the minimum mean square predictor error in that class of models. Franses and Paap (2004) stated periodic stationarity conditions for a vector autoregressive (VAR) model with periodically varying parameters, noted PVAR, in the case of four seasons and with an autoregressive order equal to one for each season. Furthermore, they discussed parameter estimation techniques. Lütkepohl (2005) studied maximum likelihood estimation of the model parameters of a general PVAR stochastic process, and test statistics for time invariance of the model coefficients are discussed. Recently, Ursu and Duchesne (2009) studied PVAR models with linear constraints on the parameters of a given season, and portmanteau test statistics for diagnosing these models were also introduced.

Seasonal and periodic time series models are quite different, since the nature of the correlation structure differs on several important aspects : purely stochastic in seasonal models and purely deterministic in periodic models. By definition, PARMA models typically do not seek to model between-period dependencies explicitly (Basawa, Lund and Shao (2004, p. 300)). In view of this, it may be tempting to exploit the desirable properties of seasonal and periodic time series. In forecasting applications, one could adjust separately a periodic autoregression and a SARMA model to the time series data : based on the work of Newbold and Granger (1974) or Winkler and Makridakis (1983), the forecasts from these two time series models could be naturally combined. However, McLeod (1993) found that combining forecasts cannot be recommended when modeling the monthly river flow data studied in Noakes, McLeod and Hipel (1985), suggesting the limits of combining forecasts with real time series data.

In order to gain flexibility, combining the seasonal and periodic time series models offers an alternative solution. Here, we introduce a multivariate seasonal

autoregressive model with periodically varying parameters, abbreviated by the acronym SPVAR, which includes the multivariate seasonal autoregressive time series model and also the multivariate periodic time series model as special cases. Combining periodic and seasonal time series models has been considered by Basawa *et al.* (2004) in the particular situation of a first-order seasonal autoregressive process with periodically varying parameters. They studied the stationarity conditions (in the periodic sense) and the limit distributions of the least squares estimators. Here, we extend the work of Basawa *et al.* (2004) in three directions. First, we generalize the first-order process to a seasonal autoregressive stochastic process with periodically varying parameters of orders p_1 and p_2, where p_1 and p_2 denote the orders in the seasonal VAR and VAR polynomials, respectively. We study the periodic stationarity conditions of that class of stochastic processes and we derive explicitly the associated theoretical autocovariance function. Second, we present asymptotic results of the least squares estimators of the model parameters for the general SPVAR model. Third, we consider diagnostic checking SPVAR time series models. Diagnostic checking was not developed in Basawa *et al.* (2004). However, from a model building point of view, it is well recognized that checking the adequacy of models appears to be a fundamental step in the time series methodology. Residual autocovariance and autocorrelation matrices from classical multivariate autoregressive moving average (VARMA) models have been found useful for diagnosing a particular adjusted model. See for example the monograph of Li (2004). Portmanteau test statistics for VARMA models have been studied by many authors, namely Hosking (1980) and Li and McLeod (1981), amongst others. In order to check SPVAR models, we derive the asymptotic distribution of the residual autocovariance and autocorrelation matrices in that class of models, under the null hypothesis of adequacy. As an useful application of these asymptotic results, portmanteau test statistics based on a fixed number of residual autocovariance matrices are proposed for diagnosing SPVAR models. We discuss the asymptotic distributions of these new test statistics and we develop modified versions with better finite-sample properties.

It should be noted that SPVAR models may be seen as special cases of PVAR models. In fact, the same is true for SVARMA models, which admit VARMA representations with particular autoregressive and moving average structures. Consequently, it is likely that several basic properties of SPVAR models could be derived from those established in the class of PVAR models. However, developments focusing on the specific formulation defining SPVAR time series models may be more informative, both from theoretical and practical perspectives. See also the discussions of Basawa and Lund (2001, p. 654) and Ursu and Duchesne (2009).

The rest of the paper is organized as follows. In Section 4.2, the new model is presented and some theoretical properties of SPVAR stochastic processes are given. Furthermore, the sample autocovariance and autocorrelation matrices are introduced. Least squares estimators are studied in Section 4.3. In particular, the asymptotic distribution of the least squares estimators is provided, which is normal. A scoring algorithm is also described. In Section 4.4, the asymptotic distributions of the residual autocovariance and autocorrelation matrices under the null hypothesis of model adequacy are derived. We describe applications for diagnostic checking based on these asymptotic results by considering portmanteau test statistics. The results of some simulation experiments are briefly discussed. Section 4.5 reports applications using a bivariate quarterly data set on travellers entering to Canada. Section 4.6 offers some concluding remarks.

4.2. The new SPVAR model and some preliminaries

4.2.1. Definition of the SPVAR(p_1, p_2) time series model

Let $\mathbf{Y} = \{\mathbf{Y}_t, t \in \mathbb{Z}\}$ be a stochastic process, where $\mathbf{Y}_t = (Y_t(1), \ldots, Y_t(d))^\top$ represents a random vector of dimension d. Let B and B^s be the usual and seasonal lag operators, respectively, where s is a predetermined value corresponding to the seasonal period (for example, $s = 4$ and $s = 12$ for quarterly and monthly data, respectively).

Definition 4.1. *The process* $\mathbf{Y}$ *is a multivariate seasonal autoregressive stochastic process with periodically varying parameters and seasonal VAR and VAR*

orders p_1 *and* p_2*, noted SPVAR*(p_1, p_2)*, if there exists* $d \times d$ *matrices* $\mathbf{\Lambda}_k(\nu) = (\Lambda_{k,ij}(\nu))_{i,j=1,\ldots,d}$*,* $k = 1, \ldots, p_1$*, and* $\mathbf{\Phi}_k(\nu) = (\Phi_{k,ij}(\nu))_{i,j=1,\ldots,d}$*,* $k = 1, \ldots, p_2$*, for* $\nu \in \{1, \ldots, s\}$*, with* $\mathbf{\Lambda}_{p_1}(\nu) \neq \mathbf{0}$ *and* $\mathbf{\Phi}_{p_2}(\nu) \neq \mathbf{0}$*,* $\nu = 1, \ldots, s$*, such that* $\{\mathbf{Y}_{ns+\nu}\}$ *satisfies the seasonal autoregressive difference equation :*

$$\mathbf{\Lambda}(B^s; \nu)\mathbf{\Phi}(B; \nu)\mathbf{Y}_{ns+\nu} = \boldsymbol{\epsilon}_{ns+\nu}, \tag{4.1}$$

where the seasonal VAR and VAR polynomials are defined by :

$$\begin{aligned} \mathbf{\Lambda}(B^s; \nu) &= \mathbf{I}_d - \mathbf{\Lambda}_1(\nu)B^s - \ldots - \mathbf{\Lambda}_{p_1}(\nu)B^{sp_1},\ \nu = 1, \ldots, s, \\ \mathbf{\Phi}(B; \nu) &= \mathbf{I}_d - \mathbf{\Phi}_1(\nu)B - \ldots - \mathbf{\Phi}_{p_2}(\nu)B^{p_2},\ \nu = 1, \ldots, s, \end{aligned}$$

respectively, and the $d \times d$ *matrix* $\mathbf{I}_d$ *denotes the identity matrix of order* d*. The error process* $\boldsymbol{\epsilon} = \{\boldsymbol{\epsilon}_t, t \in \mathbb{Z}\}$*,* $\boldsymbol{\epsilon}_t = (\epsilon_t(1), \ldots, \epsilon_t(d))^\top$*, in (4.1) corresponds to a zero mean periodic white noise, that is* $\boldsymbol{\epsilon}$ *is composed of uncorrelated random vectors, such that* $E(\boldsymbol{\epsilon}_t) = \mathbf{0}$ *and* $E(\boldsymbol{\epsilon}_{ns+\nu}\boldsymbol{\epsilon}_{ns+\nu}^\top) = \mathbf{\Sigma}_{\boldsymbol{\epsilon}}(\nu)$*, where the error covariance matrix* $\mathbf{\Sigma}_{\boldsymbol{\epsilon}}(\nu) = (\boldsymbol{\sigma}_{\boldsymbol{\epsilon},ij}(\nu))_{i,j=1,\ldots,d}$ *is assumed non singular,* $\nu = 1, \ldots, s$*.*

For fixed ν, the random vector $\mathbf{Y}_{ns+\nu}$ represents the realization during the νth season, with $\nu \in \{1, \ldots, s\}$, at year n, $n \in \mathbb{Z}$. The SPVAR process defined by (4.1) is supposed to have a zero mean, that is $E(\mathbf{Y}_t) = \mathbf{0}$. In practical applications, trends and seasonal means are first removed from the series, meaning that a model is formulated by examining $\mathbf{Y}_{ns+\nu} - \boldsymbol{\mu}_\nu$, say, where in general the mathematical expectation $E(\mathbf{Y}_{ns+\nu}) = \boldsymbol{\mu}_\nu$ may be function of the season ν.

The difference equation defining the SPVAR model is nonlinear in the parameters. However, the stochastic process (4.1) can be interpreted as a solution of the equation :

$$\mathbf{Y}_{ns+\nu} = \mathbf{\Phi}_1(\nu)\mathbf{Y}_{ns+\nu-1} + \ldots + \mathbf{\Phi}_{p_2}(\nu)\mathbf{Y}_{ns+\nu-p_2} + \mathbf{U}_{ns+\nu}, \tag{4.2}$$

where the auxiliary stochastic process $\{\mathbf{U}_t, t \in \mathbb{Z}\}$ satisfies :

$$\mathbf{U}_{ns+\nu} = \mathbf{\Lambda}_1(\nu)\mathbf{U}_{(n-1)s+\nu} + \ldots + \mathbf{\Lambda}_{p_1}(\nu)\mathbf{U}_{(n-p_1)s+\nu} + \boldsymbol{\epsilon}_{ns+\nu}. \tag{4.3}$$

In Section 4.3, which studies least squares estimation, relations (4.2) and (4.3) are used in order to find appropriate initial values in a Fisher scoring algorithm.

The SPVAR model generalizes several proposals found in the literature. Letting $\mathbf{\Lambda}_k(\nu) = \mathbf{\Lambda}_k$, $k = 1, \ldots, p_1$, and $\mathbf{\Phi}_k(\nu) = \mathbf{\Phi}_k$, $k = 1, \ldots, p_2$, for all ν in $\{1, \ldots, s\}$, that is if all model parameters are constant across the seasons, the SPVAR model reduces to the seasonal VAR stochastic process studied in Reinsel (1997, p. 219). If the seasonal parameters are set to zero, that is if $\mathbf{\Lambda}_k(\nu) = \mathbf{0}$, $k = 1, \ldots, p_1$, $\nu = 1, \ldots, s$, then a PVAR stochastic process of order p_2 in each season is obtained (see Lütkepohl (2005), amongst others). Letting $p_1 = p_2 = 1$ and $d = 1$, the SPVAR model defined by (4.1) reduces to the univariate SPAR(1,1) stochastic process introduced by Basawa *et al.* (2004). The next subsections study the stationarity conditions (in the periodic sense) and the theoretical autocovariance function of the SPVAR(p_1, p_2) time series model.

4.2.2. Causality, stationarity and theoretical autocovariance function of SPVAR stochastic processes

This section studies the fundamental properties of SPVAR stochastic processes. It is well-known that a SVARMA process can be viewed as a particular case of a VARMA process. Similarly, a SPVAR(p_1, p_2) difference equation can be formulated as a PVAR stochastic process of order $p_1 s + p_2$. For example, a SPVAR(1,1) admits the following PVAR($s + 1$) representation :

$$\mathbf{Y}_{ns+\nu} = \sum_{k=1}^{s+1} \mathbf{\Upsilon}_k(\nu) \mathbf{Y}_{ns+\nu-k} + \boldsymbol{\epsilon}_{ns+\nu},$$

where

$$\begin{aligned} \mathbf{\Upsilon}_1(\nu) &= \mathbf{\Phi}(\nu), \quad \mathbf{\Upsilon}_k(\nu) = \mathbf{0}, \quad 1 < k < s, \\ \mathbf{\Upsilon}_s(\nu) &= \mathbf{\Lambda}(\nu) \quad \text{and} \quad \mathbf{\Upsilon}_{s+1}(\nu) = -\mathbf{\Lambda}(\nu)\mathbf{\Phi}(\nu). \end{aligned}$$

It should be remarked that, in general, as the multivariate SARMA model described in Reinsel (1997), the matrix seasonal and nonseasonal VAR operators in (4.1) do not commute, and, hence, the order in which these operators are given in the difference equation will make a difference. This is clearly seen in the particular case of the SPVAR(1,1) stochastic process since in general, for a given ν, $\mathbf{\Lambda}(\nu)\mathbf{\Phi}(\nu) \neq \mathbf{\Phi}(\nu)\mathbf{\Lambda}(\nu)$.

In order to study the periodic stationarity conditions, the SPVAR(p_1, p_2) difference equation offers a companion VAR representation :

$$\mathbf{\Phi}_0^* \mathbf{Y}_n^* = \sum_{k=1}^{p^*} \mathbf{\Phi}_k^* \mathbf{Y}_{n-k}^* + \boldsymbol{\epsilon}_n^*, \tag{4.4}$$

where $\mathbf{Y}_n^* = (\mathbf{Y}_{ns+s}^\top, \mathbf{Y}_{ns+s-1}^\top, \ldots, \mathbf{Y}_{ns+1}^\top)^\top$ and $\boldsymbol{\epsilon}_n^* = (\boldsymbol{\epsilon}_{ns+s}^\top, \boldsymbol{\epsilon}_{ns+s-1}^\top, \ldots, \boldsymbol{\epsilon}_{ns+1}^\top)^\top$. The autoregressive order of the VAR stochastic process (4.4) is given by

$$p^* = \lceil (p_1 s + p_2)/s \rceil,$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to the real number x. Thus, a SPVAR(p_1, p_2) stochastic process admits a VAR($\lceil (p_1 s + p_2)/s \rceil$) representation. In the special case $d = 1$, $p_1 = p_2 = 1$, $p^* = \lceil 1 + s^{-1} \rceil = 2$, and we retrieve a result due to Basawa *et al.* (2004, p. 301) : a SPAR(1,1) difference equation can be formulated as a VAR(2) stochastic process. The $ds \times ds$ autoregressive coefficients $\mathbf{\Phi}_k^*$, $k = 0, 1, \ldots, p^*$, are given by :

$$\mathbf{\Phi}_0^* = \begin{bmatrix} \mathbf{I}_d & -\mathbf{\Phi}_1(s) & -\mathbf{\Phi}_2(s) & \ldots & -\mathbf{\Phi}_{s-2}(s) & -\mathbf{\Phi}_{s-1}(s) \\ \mathbf{0} & \mathbf{I}_d & -\mathbf{\Phi}_1(s-1) & \ldots & -\mathbf{\Phi}_{s-3}(s-1) & -\mathbf{\Phi}_{s-2}(s-1) \\ \vdots & & & \ddots & & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{I}_d & -\mathbf{\Phi}_1(2) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{I}_d \end{bmatrix},$$

and

$$\mathbf{\Phi}_k^* = - \sum_{\{i,j \in \mathbb{Z} \mid i+j=k\}} \begin{bmatrix} \mathbf{\Lambda}_i(s)\mathbf{\Phi}_{js}(s) & \mathbf{\Lambda}_i(s)\mathbf{\Phi}_{js+1}(s) & \ldots & \mathbf{\Lambda}_i(s)\mathbf{\Phi}_{js+s-1}(s) \\ \mathbf{\Lambda}_i(s-1)\mathbf{\Phi}_{js-1}(s-1) & \mathbf{\Lambda}_i(s-1)\mathbf{\Phi}_{js}(s-1) & \ldots & \mathbf{\Lambda}_i(s-1)\mathbf{\Phi}_{js+s-2}(s-1) \\ \vdots & & \ddots & \vdots \\ \mathbf{\Lambda}_i(1)\mathbf{\Phi}_{js-s+1}(1) & \mathbf{\Lambda}_i(1)\mathbf{\Phi}_{js-s+2}(1) & \ldots & \mathbf{\Lambda}_i(1)\mathbf{\Phi}_{js}(1) \end{bmatrix},$$

where we extend the definitions of $\mathbf{\Phi}_k(\nu)$ for all $k \in \mathbb{Z}$:

$$\mathbf{\Phi}_k(\nu) = \begin{cases} -\mathbf{I}_d, & k = 0, \\ \mathbf{0}, & k > p_2 \quad \text{or} \quad k < 0. \end{cases}$$

Similarly,

$$\mathbf{\Lambda}_k(\nu) = \begin{cases} -\mathbf{I}_d, & k = 0, \\ \mathbf{0}, & k > p_1. \end{cases}$$

The matrix $\mathbf{\Phi}_0^*$ pre-multiplying $\mathbf{Y}_n^*$ in (4.4) is the same than for PVAR stochastic processes. See Ursu and Duchesne (2009). It is a non-singular matrix such that $\det(\mathbf{\Phi}_0^*) = 1$, where $\det(\mathbf{A})$ stands for the determinant of the square matrix $\mathbf{A}$.

Using general properties of VAR models (see, e.g., Brockwell and Davis (1991)), it follows that the multivariate stochastic process $\{\mathbf{Y}_t^*\}$ is causal if :

$$\det\left(\mathbf{\Phi}_0^* - \mathbf{\Phi}_1^* z - \ldots - \mathbf{\Phi}_{p^*}^* z^{p^*}\right) \neq 0, \tag{4.5}$$

for all complex numbers z satisfying the condition $|z| \leq 1$. Equivalently, condition (4.5) can be expressed as :

$$\det\left(\mathbf{I}_{ds} - \mathbf{\Phi}_0^{*-1}\mathbf{\Phi}_1^* z - \ldots - \mathbf{\Phi}_0^{*-1}\mathbf{\Phi}_{p^*}^* z^{p^*}\right) \neq 0. \tag{4.6}$$

Interestingly, in the SPVAR(1,1) model, it is possible to express the condition (4.6) in function of the model parameters $\mathbf{\Phi}(\nu)$, $\mathbf{\Lambda}(\nu)$, $\nu = 1, \ldots, s$:

$$\begin{aligned}&\det\left(\mathbf{I}_{ds} - \mathbf{\Phi}_0^{*-1}\mathbf{\Phi}_1^* z - \mathbf{\Phi}_0^{*-1}\mathbf{\Phi}_2^* z^2\right) = \\ &\det\left[\prod_{\nu=1}^{s}\{\mathbf{I}_d - \mathbf{\Lambda}(\nu)z\}\right] \times \det\left\{\mathbf{I}_d - z\mathbf{\Phi}(1)\prod_{\nu=0}^{s-2}\mathbf{\Phi}(s-\nu)\right\}.\end{aligned}$$

Consequently, for a SPVAR(1,1) stochastic process, a unique causal and periodically stationary solution in the mean square sense exists provided that the eigenvalues of $\mathbf{\Phi}(1)\prod_{\nu=0}^{s-2}\mathbf{\Phi}(s-\nu)$, and those of $\mathbf{\Lambda}(\nu)$, $\nu = 1, \ldots, s$, are all strictly inferior to one in modulus. That result generalizes the causality and periodic stationary property of SPAR(1,1) stochastic process. See Basawa *et al.* (2004, p. 301) (note that in their formula (2.6) there is a small typographic mistake).

If the SPVAR(p_1, p_2) stochastic process generated by (4.1) corresponds to a causal process, it is possible to represent $\{\mathbf{Y}_{ns+\nu}\}$ through an infinite order moving average expansion :

$$\mathbf{Y}_{ns+\nu} = \sum_{k=0}^{\infty} \mathbf{\Psi}_k(\nu)\boldsymbol{\epsilon}_{ns+\nu-k}, \tag{4.7}$$

where $\mathbf{\Psi}_0(\nu) = \mathbf{I}_d$. Let $||\mathbf{A}||$ be the Euclidian norm of the matrix $\mathbf{A}$, that is $||\mathbf{A}|| = \{\mathrm{tr}(\mathbf{A}\mathbf{A}^\top)\}^{1/2}$, where $\mathrm{tr}(\mathbf{B})$ denotes the trace of the square matrix $\mathbf{B}$. The $d \times d$ matrices $\mathbf{\Psi}_k(\nu)$ can be interpreted as seasonal weights, and they satisfy

the following conditions :

$$\sum_{k=0}^{\infty} ||\mathbf{\Psi}_k(\nu)|| < \infty, \ \nu = 1, \ldots, s.$$

Using the difference equation (4.1) and the infinite order moving average expansion (4.7), it is possible to express the matrices $\mathbf{\Psi}_k(\nu)$ recursively :

$$\begin{aligned}\mathbf{\Psi}_k(\nu) &= \sum_{i=1}^{\min(k,p_2)} \mathbf{\Phi}_i(\nu)\mathbf{\Psi}_{k-i}(\nu-i) + \sum_{i=1}^{\min(k,p_1)} \mathbf{\Lambda}_i(\nu)\mathbf{\Psi}_{k-is}(\nu-is) \\ &\quad - \sum_{j=1}^{\min(k,p_1)} \sum_{i=1}^{\min(k,p_2)} \mathbf{\Lambda}_j(\nu)\mathbf{\Phi}_i(\nu)\mathbf{\Psi}_{k-i-js}(\nu-i-js), \qquad (4.8)\end{aligned}$$

where $\mathbf{\Psi}_k(\nu) = \mathbf{0}$ if $k < 0$. The notations used in (4.8) and elsewhere interpret $\mathbf{\Psi}_k(\nu)$, $\forall k \geq 0$, and $\mathbf{\Phi}_i(\nu)$, $\mathbf{\Lambda}_j(\nu)$, $\forall i, j \geq 1$ periodically in ν with period s. The recursive relations (4.8) are more involved than those established for PVAR stochastic processes. However, for $k > p_1 s + p_2$, these recursive relations rely on a finite number of terms and they remain numerically tractable as the lag order increases. See also Lund and Basawa (2000, p. 77) and Ursu and Duchesne (2009).

Using the algebraic equivalence between multivariate stationarity and periodic correlation (Gladyshev (1961), Ula (1990)), the ds-dimensional process $\{\mathbf{Y}_n^*\}$ is stationary if and only if the d-dimensional process $\{\mathbf{Y}_t\}$ is periodic stationary with period s, in the sense that :

$$\text{cov}(\mathbf{Y}_{n+s}, \mathbf{Y}_{m+s}) = \text{cov}(\mathbf{Y}_n, \mathbf{Y}_m),$$

for all integers n and m. The seasonal autocovariance function of the zero-mean process $\{\mathbf{Y}_t\}$ is defined as :

$$\mathbf{\Gamma}_{\mathbf{Y}}(h;\nu) = \text{cov}(\mathbf{Y}_{ns+\nu}, \mathbf{Y}_{ns+\nu-h}) = E(\mathbf{Y}_{ns+\nu}\mathbf{Y}_{ns+\nu-h}^{\top}),$$

which may depend on both lag h and season ν, but not on year n. The autocovariance matrix $\mathbf{\Gamma}_{\mathbf{Y}}(h;\nu)$ is interpreted periodically in ν with period s using the identity $\mathbf{\Gamma}_{\mathbf{Y}}(h;\nu) = \mathbf{\Gamma}_{\mathbf{Y}}(h;\nu+s)$. For negative lags, $\mathbf{\Gamma}_{\mathbf{Y}}(h;\nu)$ may be determined using the relation $\mathbf{\Gamma}_{\mathbf{Y}}(-h;\nu) = \mathbf{\Gamma}_{\mathbf{Y}}^{\top}(h;\nu+h)$. Since a SPVAR$(p_1, p_2)$ offers a PVAR representation, the infinite order moving average expression (4.7) gives

the following expression for the autocovariance function in terms of the seasonal weights :

$$\mathbf{\Gamma_Y}(h;\nu) = \sum_{l=0}^{\infty} \mathbf{\Psi}_{l+h}(\nu)\mathbf{\Sigma}_{\boldsymbol{\epsilon}}(\nu-l-h)\mathbf{\Psi}_l^\top(\nu-h), \tag{4.9}$$

where the covariance matrix $\mathbf{\Sigma}_{\boldsymbol{\epsilon}}(\nu)$ is interpreted periodically in ν with period s. As for VAR or PVAR stochastic processes, the autocovariance function of a SPVAR(p_1,p_2) process can be calculated recursively. For example, for $h > s+1$, the following recursive relations are easily obtained in the particular case of a SPVAR(1,1) process :

$$\mathbf{\Gamma_Y}(h;\nu) = \mathbf{\Phi}(\nu)\mathbf{\Gamma_Y}(h-1;\nu-1)+\mathbf{\Lambda}(\nu)\mathbf{\Gamma_Y}(h-s;\nu)-\mathbf{\Lambda}(\nu)\mathbf{\Phi}(\nu)\mathbf{\Gamma_Y}(h-s-1;\nu-1). \tag{4.10}$$

Similar but more tedious calculations yield the recursive relations for the theoretical autocovariance matrices of a SPVAR(p_1,p_2) stochastic process. In the next section, empirical versions of the autocovariance and autocorrelation matrices are presented.

4.2.3. Sample autocovariance and autocorrelation matrices

Let

$$\boldsymbol{\beta}(\nu) = (\mathrm{vec}^\top\{\mathbf{\Phi}_1(\nu)\},\ldots,\mathrm{vec}^\top\{\mathbf{\Phi}_{p_2}(\nu)\},\mathrm{vec}^\top\{\mathbf{\Lambda}_1(\nu)\},\ldots,\mathrm{vec}^\top\{\mathbf{\Lambda}_{p_1}(\nu)\})^\top, \tag{4.11}$$

$\nu = 1,\ldots,s$, be $(p_1+p_2)d^2 \times 1$ vectors corresponding to the model parameters of a SPVAR(p_1,p_2) time series model, where vec$(\mathbf{A})$ corresponds to the vector obtained by stacking the columns of $\mathbf{A}$ (see Harville (1997, Chapter 16.3)). For any particular vector $\dot{\boldsymbol{\beta}}(\nu)$, we introduce the model residuals :

$$\dot{\boldsymbol{\epsilon}}_{ns+\nu} = \begin{cases} \dot{\mathbf{\Lambda}}(B^s;\nu)\dot{\mathbf{\Phi}}(B;\nu)\mathbf{Y}_{ns+\nu}, & ns+\nu > p_1 s + p_2, \\ \mathbf{0}, & ns+\nu \leq p_1 s + p_2, \end{cases}$$

where $\dot{\mathbf{\Lambda}}(B^s;\nu) = \mathbf{I}_d - \ldots - \dot{\mathbf{\Lambda}}_{p_1}(\nu)B^{sp_1}$ and $\dot{\mathbf{\Phi}}(B;\nu) = \mathbf{I}_d - \ldots - \dot{\mathbf{\Phi}}_{p_2}(\nu)B^{p_2}$, which are well-defined for $n = 0,1,\ldots,N-1$, where Ns denotes the sample size. Let $\mathbf{\Gamma}_{\boldsymbol{\epsilon}}(h;\nu) = \mathrm{cov}(\boldsymbol{\epsilon}_{ns+\nu},\boldsymbol{\epsilon}_{ns+\nu-h})$ and $\boldsymbol{\rho}_{\boldsymbol{\epsilon}}(h,\nu) = \mathbf{\Gamma}_0^{-1}(\nu)\mathbf{\Gamma}_{\boldsymbol{\epsilon}}(h;\nu)\mathbf{\Gamma}_0^{-1}(\nu-h)$ be the lag h theoretical autocovariance and autocorrelation matrices at season ν of

the error process $\boldsymbol{\epsilon}$, respectively, where $\boldsymbol{\Gamma}_0(\nu) = \text{diag}(\boldsymbol{\sigma}^{1/2}_{\boldsymbol{\epsilon},11}(\nu), \ldots, \boldsymbol{\sigma}^{1/2}_{\boldsymbol{\epsilon},dd}(\nu))$. We introduce the sample autocovariance matrices $\mathbf{C}_{\dot{\boldsymbol{\epsilon}}}(h;\nu) = \left(C_{\dot{\boldsymbol{\epsilon}},ij}(h;\nu)\right)_{i,j=1,\ldots,d}$:

$$\mathbf{C}_{\dot{\boldsymbol{\epsilon}}}(h;\nu) = \begin{cases} N^{-1}\sum_{n=h}^{N-1} \dot{\boldsymbol{\epsilon}}_{ns+\nu}\dot{\boldsymbol{\epsilon}}^{\top}_{ns+\nu-h}, & h \geq 0, \\ \mathbf{C}^{\top}_{\dot{\boldsymbol{\epsilon}}}(-h;\nu-h), & h < 0. \end{cases}$$

Let $\mathbf{c}_{\dot{\boldsymbol{\epsilon}}}(h;\nu) = \text{vec}\{\mathbf{C}_{\dot{\boldsymbol{\epsilon}}}(h;\nu)\}$. The vector of sample autocovariances are collected in the following random vector :

$$\mathbf{c}_{\dot{\boldsymbol{\epsilon}}}(\nu) = (\mathbf{c}^{\top}_{\dot{\boldsymbol{\epsilon}}}(1;\nu), \ldots, \mathbf{c}^{\top}_{\dot{\boldsymbol{\epsilon}}}(M;\nu))^{\top}, \tag{4.12}$$

where the maximal lag order M represents a fixed integer with respect to the number of years N, satisfying the relation $1 \leq M < N$. Similarly, the vector of sample autocorrelations is given by $\mathbf{r}_{\dot{\boldsymbol{\epsilon}}}(\nu) = (\mathbf{r}^{\top}_{\dot{\boldsymbol{\epsilon}}}(1;\nu), \ldots, \mathbf{r}^{\top}_{\dot{\boldsymbol{\epsilon}}}(M;\nu))^{\top}$, where the lag h sample autocorrelation matrix satisfies the relations :

$$\begin{aligned} \mathbf{r}_{\dot{\boldsymbol{\epsilon}}}(h;\nu) &= \text{vec}\left\{\mathbf{D}^{-1}_{\dot{\boldsymbol{\epsilon}}}(\nu)\mathbf{C}_{\dot{\boldsymbol{\epsilon}}}(h;\nu)\mathbf{D}^{-1}_{\dot{\boldsymbol{\epsilon}}}(\nu-h)\right\}, \\ &= \left(\mathbf{D}^{-1}_{\dot{\boldsymbol{\epsilon}}}(\nu-h) \otimes \mathbf{D}^{-1}_{\dot{\boldsymbol{\epsilon}}}(\nu)\right)\mathbf{c}_{\dot{\boldsymbol{\epsilon}}}(h;\nu), \end{aligned}$$

where '$\otimes$' denotes the Kronecker product, with

$$\mathbf{D}_{\dot{\boldsymbol{\epsilon}}}(\nu) = \text{diag}\left(\mathbf{C}^{1/2}_{\dot{\boldsymbol{\epsilon}},11}(0;\nu), \ldots, \mathbf{C}^{1/2}_{\dot{\boldsymbol{\epsilon}},dd}(0;\nu)\right).$$

In Section 4.3, the asymptotic properties of the least squares estimators are discussed, and in Section 4.4, the asymptotic distributions of the residual autocovariance and autocorrelation vectors $\mathbf{c}_{\dot{\boldsymbol{\epsilon}}}(h;\nu)$ and $\mathbf{r}_{\dot{\boldsymbol{\epsilon}}}(h;\nu)$, $h = 1, \ldots, M$, are established, where $\hat{\boldsymbol{\epsilon}}_{ns+\nu}$, $n = 0, 1, \ldots, N-1$, correspond to the least squares residuals, $\nu = 1, \ldots, s$.

4.3. Least squares estimation of the model parameters

4.3.1. Asymptotic properties of the least squares estimators

In this section, we study the asymptotic properties of least squares estimators from a causal SPVAR model. Consider the time series data $\mathbf{Y}_{ns+\nu}$, $n = 0, 1, \ldots, N-1$, $\nu = 1, \ldots, s$, with sample size Ns. The least squares estimators

of $\mathbf{\Lambda}_k(\nu)$, $k = 1, \ldots, p_1$, and of $\mathbf{\Phi}_k(\nu)$, $k = 1, \ldots, p_2$, are obtained by minimizing the least squares criterion :

$$S_{LS} \equiv S_{LS}\{\boldsymbol{\beta}(1), \ldots, \boldsymbol{\beta}(s); \mathbf{\Sigma}_{\boldsymbol{\epsilon}}(1), \ldots, \mathbf{\Sigma}_{\boldsymbol{\epsilon}}(s)\} = \sum_{\nu=1}^{s} S\{\boldsymbol{\beta}(\nu); \mathbf{\Sigma}_{\boldsymbol{\epsilon}}(\nu)\}, \quad (4.13)$$

where $S\{\boldsymbol{\beta}(\nu); \mathbf{\Sigma}_{\boldsymbol{\epsilon}}(\nu)\} = \sum_{n=0}^{N-1} \boldsymbol{\epsilon}_{ns+\nu}^{\top} \mathbf{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\nu) \boldsymbol{\epsilon}_{ns+\nu}$. By using classical results on matrix differentiation (see, e.g., Harville (1997, Chap. 15) or Lütkepohl (2005, Section A.13)), differentiating (4.13) with respect to $\mathbf{\Lambda}_k(\nu)$, $k = 1, \ldots, p_1$, and to $\mathbf{\Phi}_k(\nu)$, $k = 1, \ldots, p_2$, allow us to show that the least squares estimators must satisfy the following system :

$$\begin{aligned}
\frac{\partial S_{LS}}{\partial \mathrm{vec}^{\top}\{\mathbf{\Lambda}_k(\nu)\}} &= 2 \sum_{n=0}^{N-1} \boldsymbol{\epsilon}_{ns+\nu}^{\top} \mathbf{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\nu) \frac{\partial \boldsymbol{\epsilon}_{ns+\nu}}{\partial \mathrm{vec}^{\top}\{\mathbf{\Lambda}_k(\nu)\}} = \mathbf{0}, \; k = 1, \ldots, p_1, \\
\frac{\partial S_{LS}}{\partial \mathrm{vec}^{\top}\{\mathbf{\Phi}_k(\nu)\}} &= 2 \sum_{n=0}^{N-1} \boldsymbol{\epsilon}_{ns+\nu}^{\top} \mathbf{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\nu) \frac{\partial \boldsymbol{\epsilon}_{ns+\nu}}{\partial \mathrm{vec}^{\top}\{\mathbf{\Phi}_k(\nu)\}} = \mathbf{0}, \; k = 1, \ldots, p_2,
\end{aligned}$$

$\nu \in \{1, \ldots, s\}$, where the derivatives correspond to the following $d \times d^2$ matrices :

$$\begin{aligned}
\frac{\partial \boldsymbol{\epsilon}_{ns+\nu}}{\partial \mathrm{vec}^{\top}\{\mathbf{\Lambda}_k(\nu)\}} &= -\{\mathbf{Y}_{(n-k)s+\nu}^{\top} \otimes \mathbf{I}_d\} + \{\mathbf{Y}_{(n-k)s+\nu-1}^{\top} \mathbf{\Phi}_1^{\top}(\nu) \otimes \mathbf{I}_d\} + \ldots + \\
&\quad \{\mathbf{Y}_{(n-k)s+\nu-p_2}^{\top} \mathbf{\Phi}_{p_2}^{\top}(\nu) \otimes \mathbf{I}_d\}, \; k = 1, \ldots, p_1, \quad (4.14) \\
\frac{\partial \boldsymbol{\epsilon}_{ns+\nu}}{\partial \mathrm{vec}^{\top}\{\mathbf{\Phi}_k(\nu)\}} &= -\{\mathbf{Y}_{ns+\nu-k}^{\top} \otimes \mathbf{I}_d\} + \{\mathbf{Y}_{(n-1)s+\nu-k}^{\top} \otimes \mathbf{\Lambda}_1(\nu)\} + \ldots + \\
&\quad \{\mathbf{Y}_{(n-p_1)s+\nu-k}^{\top} \otimes \mathbf{\Lambda}_{p_1}(\nu)\}, \; k = 1, \ldots, p_2. \quad (4.15)
\end{aligned}$$

The derivatives (4.14) and (4.15) are obtained by first vectorizing the SPVAR difference equation (4.1), and then using the well-known relations

$$\mathrm{vec}(\mathbf{ABC}) = (\mathbf{C}^{\top} \otimes \mathbf{A})\mathrm{vec}(\mathbf{B})$$

and

$$\partial \mathrm{vec}\{\mathbf{AB}\}/\partial \boldsymbol{\beta}^{\top} = (\mathbf{I}_q \otimes \mathbf{A}) \partial \mathrm{vec}(\mathbf{B})/\partial \boldsymbol{\beta}^{\top} + (\mathbf{B}^{\top} \otimes \mathbf{I}_n) \partial \mathrm{vec}(\mathbf{A})/\partial \boldsymbol{\beta}^{\top},$$

where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are matrices with dimensions $n \times p$, $p \times q$ and $q \times r$, respectively. It appears useful to introduce the random vector $\mathbf{Z}_{ns+\nu} = \mathbf{\Delta}_{ns+\nu}^{\top} \mathbf{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\nu) \boldsymbol{\epsilon}_{ns+\nu}$ of dimension $(p_1 + p_2)d^2$, where :

$$\mathbf{\Delta}_{ns+\nu} = \frac{\partial \boldsymbol{\epsilon}_{ns+\nu}}{\partial \boldsymbol{\beta}^{\top}(\nu)}. \quad (4.16)$$

The matrix of derivatives (4.16) is of dimension $d\times(p_1+p_2)d^2$. Letting $\mathbf{S}_N\{\boldsymbol{\beta}(\nu)\} = \sum_{n=0}^{N-1}\mathbf{Z}_{ns+\nu}$, the first-order conditions can be expressed simply as $\mathbf{S}_N\{\boldsymbol{\beta}(\nu)\} = \mathbf{0}$, $\nu = 1,\ldots,s$. Proposition 4.1 is needed in order to establish the asymptotic behavior of the least squares estimators $\hat{\boldsymbol{\beta}}(\nu)$ of $\boldsymbol{\beta}(\nu)$. The symbols '$\xrightarrow{d}$' and '$\xrightarrow{p}$' stand for convergence in distribution and probability, respectively, and $N_d(\boldsymbol{\mu},\boldsymbol{\Sigma})$ denotes a d-dimensional normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$.

Proposition 4.1. *Let a stochastic process*

$$\mathbf{Y} = \{\mathbf{Y}_t, t\in\mathbb{Z}\},\ \mathbf{Y}_t = (Y_t(1),\ldots,Y_t(d))^\top,$$

be generated by the SPVAR difference equation (4.1), and presume that the causality condition given by (4.5) is satisfied. Suppose that the error term $\boldsymbol{\epsilon} = \{\boldsymbol{\epsilon}_t\}$ corresponds to a periodic white noise composed of independent random vectors satisfying $E(\boldsymbol{\epsilon}_{ns+\nu}) = \mathbf{0}$ and $\mathrm{var}(\boldsymbol{\epsilon}_{ns+\nu}) = \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu)$, and in addition assume that the fourth-order moments of $\boldsymbol{\epsilon}_t$ are finite :

$$E\{|\epsilon_t(i)\epsilon_t(j)\epsilon_t(k)\epsilon_t(l)|\} < \infty,\ \forall i,j,k,l = 1,\ldots,d;\ \forall t\in\mathbb{Z}.$$

Then :

$$N^{-1}\mathbf{S}_N\{\boldsymbol{\beta}(\nu)\} \xrightarrow{p} \mathbf{0}, \tag{4.17}$$

$$N^{-1}\sum_{n=0}^{N-1}\mathbf{Z}_{ns+\nu}\mathbf{Z}_{ns+\nu}^\top \xrightarrow{p} \boldsymbol{\Omega}(\nu), \tag{4.18}$$

$$N^{-1/2}\mathbf{S}_N\{\boldsymbol{\beta}(\nu)\} \xrightarrow{d} N_{(p_1+p_2)d^2}(\mathbf{0},\boldsymbol{\Omega}(\nu)), \tag{4.19}$$

$$N^{-1}\frac{\partial\mathbf{S}_N\{\boldsymbol{\beta}(\nu)\}}{\partial\boldsymbol{\beta}^\top(\nu)} \xrightarrow{p} \boldsymbol{\Omega}(\nu), \tag{4.20}$$

where the matrix $\boldsymbol{\Omega}(\nu)$ corresponds to the $(p_1+p_2)d^2\times(p_1+p_2)d^2$ matrix :

$$\boldsymbol{\Omega}(\nu) = E\{\boldsymbol{\Delta}_{ns+\nu}^\top\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\nu)\boldsymbol{\Delta}_{ns+\nu}\}, \tag{4.21}$$

and $\boldsymbol{\Delta}_{ns+\nu}$ is defined by (4.16).

PROOF. Let $\mathcal{F}_{ns+\nu} = \sigma(\mathbf{Y}_{ns+\nu},\mathbf{Y}_{ns+\nu-1},\ldots)$ be the sigma-algebra associated with the random vectors $\{\mathbf{Y}_{ns+\nu-k}, k\geq 0\}$. Given the independence assumption of the error term $\{\boldsymbol{\epsilon}_t\}$, it follows that $\boldsymbol{\epsilon}_{ns+\nu}$ is independent of $\mathbf{Y}_{ns+\nu-1},\mathbf{Y}_{ns+\nu-2},\ldots,$

$\nu = 1, \ldots, s$. Since $E(\boldsymbol{\epsilon}_{ns+\nu}) = \mathbf{0}$, the conditional expectation of the random vector $\mathbf{Z}_{ns+\nu}$ with respect to $\mathcal{F}_{ns+\nu-1}$ is the null vector :

$$E\left[\mathbf{Z}_{ns+\nu}^{\top}|\mathcal{F}_{ns+\nu-1}\right] = E(\boldsymbol{\epsilon}_{ns+\nu}^{\top})E\left[\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\nu)\boldsymbol{\Delta}_{ns+\nu}|\mathcal{F}_{ns+\nu-1}\right] = \mathbf{0}^{\top}.$$

Consequently, $\{\mathbf{Z}_{ns+\nu}\}$ represents a martingale difference sequence. The unconditional covariance matrix of $\mathbf{Z}_{ns+\nu}$ is given by $E(\mathbf{Z}_{ns+\nu}\mathbf{Z}_{ns+\nu}^{\top}) = \boldsymbol{\Omega}(\nu)$.
Invoking the law of large numbers for martingale difference sequences (see, e.g., Hamilton (1994, Chap. 7) or White (2001, Chap. 3)), it may be easily shown that $N^{-1}\mathbf{S}_N\{\boldsymbol{\beta}(\nu)\} \xrightarrow{p} \mathbf{0}$ and also $N^{-1}\sum_{n=0}^{N-1}\mathbf{Z}_{ns+\nu}\mathbf{Z}_{ns+\nu}^{\top} \xrightarrow{p} \boldsymbol{\Omega}(\nu)$. These results establish relations (4.17) and (4.18). Furthermore, the central limit theorem for martingale difference sequences (see, e.g., Hamilton (1994, Chap. 7) or White (2001, Chap. 5)) gives the asymptotic normal distribution of $N^{-1/2}\mathbf{S}_N\{\boldsymbol{\beta}(\nu)\}$, and hence relation (4.19). In order to show (4.20), Proposition A.2 from Lütkepohl (2005, p. 666) yields :

$$\begin{aligned}\frac{\partial \mathbf{S}_N\{\boldsymbol{\beta}(\nu)\}}{\partial \boldsymbol{\beta}^{\top}(\nu)} &= \sum_{n=0}^{N-1}\frac{\partial \mathrm{vec}\{\boldsymbol{\Delta}_{ns+\nu}^{\top}\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\nu)\boldsymbol{\epsilon}_{ns+\nu}\}}{\partial \boldsymbol{\beta}^{\top}(\nu)},\\ &\quad \sum_{n=0}^{N-1}\left\{\boldsymbol{\Delta}_{ns+\nu}^{\top}\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\nu)\frac{\partial \boldsymbol{\epsilon}_{ns+\nu}}{\partial \boldsymbol{\beta}^{\top}(\nu)} + (\boldsymbol{\epsilon}_{ns+\nu}^{\top}\otimes \mathbf{I}_d)\frac{\partial \mathrm{vec}\{\boldsymbol{\Delta}_{ns+\nu}^{\top}\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\nu)\}}{\partial \boldsymbol{\beta}^{\top}(\nu)}\right\}.\end{aligned}$$

Another application of the law of large numbers for martingale difference sequences offers :

$$N^{-1}\sum_{n=0}^{N-1}\boldsymbol{\Delta}_{ns+\nu}^{\top}\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\nu)\boldsymbol{\Delta}_{ns+\nu} \xrightarrow{p} \boldsymbol{\Omega}(\nu),$$

which shows relation (4.20). This proves Proposition 4.1. □

The asymptotic covariance matrix $\boldsymbol{\Omega}(\nu)$ of $N^{-1/2}\mathbf{S}_N\{\boldsymbol{\beta}(\nu)\}$ appears in the asymptotic distribution of the least squares estimators and in the asymptotic distribution of the residual autocovariance and autocorrelation matrices; it is given explicitly in Section 4.3.3 for a SPVAR(1,1) model. Theorem 4.1 states the asymptotic distribution of the least squares estimators $\hat{\boldsymbol{\beta}}(\nu)$ of $\boldsymbol{\beta}(\nu)$, $\nu = 1, \ldots, s$.

Theorem 4.1. *Let a stochastic process* $\mathbf{Y} = \{\mathbf{Y}_t, t \in \mathbb{Z}\}$, $\mathbf{Y}_t = (Y_t(1), \ldots, Y_t(d))^{\top}$, *be generated by a causal SPVAR*(p_1, p_2) *model. Let the parameter models* $\boldsymbol{\beta}(\nu)$ *be defined by (4.11),* $\nu = 1, \ldots, s$. *Under the conditions of Proposition 4.1, the distributions of the least squares estimators* $\hat{\boldsymbol{\beta}}(\nu)$ *of* $\boldsymbol{\beta}(\nu)$, $\nu = 1, \ldots, s$, *obtained by*

minimizing (4.13), are normal asymptotically, that is :

$$N^{1/2}\{\hat{\boldsymbol{\beta}}(\nu) - \boldsymbol{\beta}(\nu)\} \xrightarrow{d} N_{(p_1+p_2)d^2}\left(\mathbf{0}, \boldsymbol{\Omega}^{-1}(\nu)\right), \ \nu = 1, \ldots, s. \tag{4.22}$$

The matrix $\boldsymbol{\Omega}(\nu)$ is defined in Proposition 4.1. Furthermore, $N^{1/2}\{\hat{\boldsymbol{\beta}}(\nu) - \boldsymbol{\beta}(\nu)\}$ and $N^{1/2}\{\hat{\boldsymbol{\beta}}(\nu') - \boldsymbol{\beta}(\nu')\}$ are asymptotically independent, $\nu \neq \nu'$, $\nu, \nu' = 1, \ldots, s$.

PROOF. Using the same kind of arguments that those found in Theorem 8.4.1 in Fuller (1996, p. 432), the convergence in probability is easily found, that is $\hat{\boldsymbol{\beta}}(\nu) \xrightarrow{p} \boldsymbol{\beta}(\nu)$. See also Basawa and Lund (2001, p. 658) for a similar argument. A Taylor series expansion of $\mathbf{S}_N\{\boldsymbol{\beta}(\nu)\}$ around $\boldsymbol{\beta}(\nu)$ and evaluating at the point $\hat{\boldsymbol{\beta}}(\nu)$ gives the first order terms :

$$\mathbf{S}_N\{\hat{\boldsymbol{\beta}}(\nu)\} = \mathbf{S}_N\{\boldsymbol{\beta}(\nu)\} + \frac{\partial \mathbf{S}_N\{\boldsymbol{\beta}(\nu)\}}{\partial \boldsymbol{\beta}^\top(\nu)}\{\hat{\boldsymbol{\beta}}(\nu) - \boldsymbol{\beta}(\nu)\} + \mathbf{O}_p(1). \tag{4.23}$$

Since $\hat{\boldsymbol{\beta}}(\nu)$ represents the least squares solution, $\mathbf{S}_N\{\hat{\boldsymbol{\beta}}(\nu)\} = \mathbf{0}$, and from that observation we obtain :

$$\mathbf{0} = N^{-1/2}\mathbf{S}_N\{\boldsymbol{\beta}(\nu)\} + \left[N^{-1}\frac{\partial \mathbf{S}_N\{\boldsymbol{\beta}(\nu)\}}{\partial \boldsymbol{\beta}^\top(\nu)}\right] N^{1/2}\{\hat{\boldsymbol{\beta}}(\nu) - \boldsymbol{\beta}(\nu)\} + \mathbf{o}_p(1). \tag{4.24}$$

Results (4.19) and (4.20) in Proposition 4.1, and an application of Slutsky theorem allow us to show (4.22). The joint asymptotic normality of

$$N^{1/2}\{\hat{\boldsymbol{\beta}}^\top(1) - \boldsymbol{\beta}^\top(1), \ldots, \hat{\boldsymbol{\beta}}^\top(s) - \boldsymbol{\beta}^\top(s)\}^\top$$

follows using the same kind of manipulations that those for a single season ν, and from this the asymptotic independence between $N^{1/2}\{\hat{\boldsymbol{\beta}}(\nu) - \boldsymbol{\beta}(\nu)\}$ and $N^{1/2}\{\hat{\boldsymbol{\beta}}(\nu') - \boldsymbol{\beta}(\nu')\}$, $\nu \neq \nu'$ is easily deduced. This concludes the proof of Theorem 4.1. □

Theorem 4.1 generalizes Theorem 3.1 in Basawa *et al.* (2004), which is obtained by setting $d = 1$ and $p_1 = p_2 = 1$. In order to estimate the asymptotic covariance matrix of the least squares estimators, a consistent estimator of $\boldsymbol{\Omega}(\nu)$ is obtained by estimating the theoretical autocovariance matrices by the sample autocovariance matrices. Furthermore, a consistent estimator of the covariance

matrix $\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu)$ is given by :

$$\hat{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon}}(\nu) = N^{-1} \sum_{n=0}^{N-1} \hat{\boldsymbol{\epsilon}}_{ns+\nu} \hat{\boldsymbol{\epsilon}}_{ns+\nu}^{\top}, \ \nu = 1, \ldots, s, \tag{4.25}$$

where $\hat{\boldsymbol{\epsilon}}_{ns+\nu}$, $n = 0, 1, \ldots, N-1$, denote the least squares residuals.

4.3.2. Computation of the model parameters in SPVAR time series models

The least squares system for SPVAR models is nonlinear in the parameters. It can be solved using a Fisher scoring algorithm. In order to find reasonable initial values of $\boldsymbol{\Lambda}_k(\nu)$, $k = 1, \ldots, p_1$, and of $\boldsymbol{\Phi}_k(\nu)$, $k = 1, \ldots, p_2$, the system of equations composed of (4.2) and (4.3) suggests a two-step regression method. The algorithm goes as follows :

Step 1: In order to obtain initial estimators $\boldsymbol{\Phi}_1^{(0)}(\nu), \ldots, \boldsymbol{\Phi}_{p_2}^{(0)}(\nu)$, the sum $\sum_{n=0}^{N-1} \mathbf{U}_{ns+\nu}^{\top} \mathbf{U}_{ns+\nu}$ is minimized with respect to $\boldsymbol{\Phi}_1(\nu), \ldots, \boldsymbol{\Phi}_{p_2}(\nu)$, where $\{\mathbf{U}_{ns+\nu}\}$ represents the auxiliary stochastic process defined by (4.3). Let $\boldsymbol{\Phi}_1^{(0)}(\nu), \ldots, \boldsymbol{\Phi}_{p_2}^{(0)}(\nu)$ be the solutions of the system

$$\sum_{n=0}^{N-1} \mathbf{U}_{ns+\nu}^{\top} \partial \mathbf{U}_{ns+\nu} / \partial \mathrm{vec}^{\top} \boldsymbol{\Phi}_k(\nu) = \mathbf{0}^{\top},$$

$k = 1, \ldots, p_2$. For example, in the particular case of a SPVAR(1,1) model, we obtain easily the following solution :

$$\boldsymbol{\Phi}^{(0)}(\nu) = \left(\sum_{n=0}^{N-1} \mathbf{Y}_{ns+\nu} \mathbf{Y}_{ns+\nu-1}^{\top} \right) \left(\sum_{n=0}^{N-1} \mathbf{Y}_{ns+\nu-1} \mathbf{Y}_{ns+\nu-1}^{\top} \right)^{-1}.$$

Step 2: Based on the initial estimators calculated in Step 1, the following residuals are calculated :

$$\hat{\mathbf{U}}_{ns+\nu} = \mathbf{Y}_{ns+\nu} - \boldsymbol{\Phi}_1^{(0)}(\nu) \mathbf{Y}_{ns+\nu-1} - \ldots - \boldsymbol{\Phi}_{p_2}^{(0)}(\nu) \mathbf{Y}_{ns+\nu-p_2},$$

$n = 0, \ldots, N-1$. Then, the sum $\sum_{n=0}^{N-1} \boldsymbol{\epsilon}_{ns+\nu}^{\top} \boldsymbol{\epsilon}_{ns+\nu}$ is minimized with respect to $\boldsymbol{\Lambda}_1(\nu), \ldots, \boldsymbol{\Lambda}_{p_1}(\nu)$. For example, the estimator of $\boldsymbol{\Lambda}(\nu)$ in the SPVAR(1,1) is given by :

$$\boldsymbol{\Lambda}^{(0)}(\nu) = \left(\sum_{n=0}^{N-1} \hat{\mathbf{U}}_{ns+\nu} \hat{\mathbf{U}}_{(n-1)s+\nu}^{\top} \right) \left(\sum_{n=0}^{N-1} \hat{\mathbf{U}}_{(n-1)s+\nu} \hat{\mathbf{U}}_{(n-1)s+\nu}^{\top} \right)^{-1}.$$

Step 3: Steps 1 and 2 provide the initial values of $\boldsymbol{\beta}(\nu)$, $\nu = 1, \ldots, s$:

$$\begin{aligned}\hat{\boldsymbol{\beta}}^{(0)}(\nu) &= (\mathrm{vec}^\top\{\boldsymbol{\Phi}_1^{(0)}(\nu)\}, \ldots, \mathrm{vec}^\top\{\boldsymbol{\Phi}_{p_2}^{(0)}(\nu)\}, \mathrm{vec}^\top\{\boldsymbol{\Lambda}_1^{(0)}(\nu)\}, \ldots, \\ &\quad \mathrm{vec}^\top\{\boldsymbol{\Lambda}_{p_1}^{(0)}(\nu)\})^\top, \nu = 1, \ldots, s.\end{aligned}$$

Then, a scoring algorithm can be realized to obtain the least squares estimators $\hat{\boldsymbol{\beta}}(\nu)$ of $\boldsymbol{\beta}(\nu)$, $\nu = 1, \ldots, s$. An asymptotic version of relations (4.23) and (4.24) is given by :

$$\hat{\boldsymbol{\beta}}(\nu) - \boldsymbol{\beta}(\nu) = \boldsymbol{\Omega}^{-1}(\nu) N^{-1} \mathbf{S}_N\{\boldsymbol{\beta}(\nu)\} + \mathbf{o}_P(N^{-1/2}). \tag{4.26}$$

The least squares estimators can be computed iteratively using the following iterative scheme :

$$\hat{\boldsymbol{\beta}}^{(i+1)}(\nu) = \hat{\boldsymbol{\beta}}^{(i)}(\nu) + \{\hat{\boldsymbol{\Omega}}^{(i)}(\nu)\}^{-1} N^{-1} \mathbf{S}_N\{\hat{\boldsymbol{\beta}}^{(i)}(\nu)\}, \tag{4.27}$$

where $\hat{\boldsymbol{\Omega}}^{(i)}(\nu)$ represents an estimator of $\boldsymbol{\Omega}(\nu)$ based on $\hat{\boldsymbol{\beta}}^{(i)}(\nu)$ at iteration i. The iterative steps are repeated until convergence is reached.

The scoring algorithm described in this section has been implemented using the MATLAB software and is available by communicating directly with the authors.

4.3.3. Example : calculation of $\boldsymbol{\Omega}(\nu)$ in the SPVAR(1,1) time series model

It may be informative to give explicit expressions of $\boldsymbol{\Omega}(\nu)$ for the SPVAR(1,1) time series model, which can be useful for implementing the scoring algorithm described in the previous section. In that particular case, the $2d^2 \times 2d^2$ covariance matrix $\boldsymbol{\Omega}(\nu)$ may be written as the 2×2 block matrix :

$$\boldsymbol{\Omega}(\nu) = \begin{pmatrix} \boldsymbol{\Omega}_{11}(\nu) & \boldsymbol{\Omega}_{12}(\nu) \\ \boldsymbol{\Omega}_{21}(\nu) & \boldsymbol{\Omega}_{22}(\nu) \end{pmatrix},$$

where $\boldsymbol{\Omega}_{12}(\nu) = \{\boldsymbol{\Omega}_{21}(\nu)\}^\top$. Using the basic properties of the autocovariance function $\boldsymbol{\Gamma}_{\mathbf{Y}}(h; \nu)$, explicit expressions for the $d^2 \times d^2$ matrices $\boldsymbol{\Omega}_{11}(\nu)$, $\boldsymbol{\Omega}_{12}(\nu)$

and $\boldsymbol{\Omega}_{22}(\nu)$ are given by :

$$\begin{aligned}
\boldsymbol{\Omega}_{11}(\nu) &= \boldsymbol{\Gamma}_{\mathbf{Y}}(0;\nu-1)\otimes\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\nu)-\boldsymbol{\Gamma}_{\mathbf{Y}}^{\top}(s;\nu-1)\otimes\boldsymbol{\Lambda}^{\top}(\nu)\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\nu)- \\
&\quad \boldsymbol{\Gamma}_{\mathbf{Y}}(s;\nu-1)\otimes\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\nu)\boldsymbol{\Lambda}(\nu)+\boldsymbol{\Gamma}_{\mathbf{Y}}(0;\nu-1)\otimes\boldsymbol{\Lambda}^{\top}(\nu)\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\nu)\boldsymbol{\Lambda}(\nu), \\
\boldsymbol{\Omega}_{12}(\nu) &= \boldsymbol{\Gamma}_{\mathbf{Y}}(s-1;\nu-1)\otimes\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\nu)-\boldsymbol{\Gamma}_{\mathbf{Y}}(1;\nu)\otimes\boldsymbol{\Lambda}^{\top}(\nu)\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\nu)- \\
&\quad \boldsymbol{\Gamma}_{\mathbf{Y}}(s;\nu-1)\boldsymbol{\Phi}^{\top}(\nu)\otimes\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\nu)+\boldsymbol{\Gamma}_{\mathbf{Y}}(0;\nu-1)\boldsymbol{\Phi}^{\top}(\nu)\otimes\boldsymbol{\Lambda}^{\top}(\nu)\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\nu), \\
\boldsymbol{\Omega}_{22}(\nu) &= \boldsymbol{\Gamma}_{\mathbf{Y}}(0;\nu)\otimes\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\nu)-\boldsymbol{\Phi}(\nu)\boldsymbol{\Gamma}_{\mathbf{Y}}^{\top}(1;\nu)\otimes\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\nu)- \\
&\quad \boldsymbol{\Gamma}_{\mathbf{Y}}(1;\nu)\boldsymbol{\Phi}^{\top}(\nu)\otimes\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\nu)+\boldsymbol{\Phi}(\nu)\boldsymbol{\Gamma}_{\mathbf{Y}}(0;\nu-1)\boldsymbol{\Phi}^{\top}(\nu)\otimes\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\nu).
\end{aligned}$$

For making inference about the model parameters $\boldsymbol{\beta}(\nu)$, $\nu=1,\ldots,s$, the matrix $\boldsymbol{\Omega}(\nu)$ can be consistently estimated by replacing $\boldsymbol{\Phi}(\nu)$ and $\boldsymbol{\Lambda}(\nu)$ by their corresponding least squares estimators, by estimating the theoretical autocovariance matrices by sample versions, and by considering (4.25) as a consistent estimator of the variance of the error term $\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu)$, for $\nu=1,\ldots,s$.

4.4. Asymptotic distributions of the residual autocovariance and autocorrelation matrices

Basawa *et al.* (2004) did not discuss model checking SPAR time series models. It is the purpose of this section to test the null hypothesis of adequacy of multivariate SPVAR models :

$$H_0 : \mathbf{c}_{\boldsymbol{\epsilon}}(l;\nu)=\mathbf{0},\ l=1,2,\ldots.$$

In the class of PVAR time series models of order p for each season, Ursu and Duchesne (2009) studied the asymptotic distribution of the portmanteau test statistics :

$$\mathcal{Q}_M(\nu) = N\sum_{l=1}^{M}\mathrm{tr}\{\mathbf{C}_{\hat{\boldsymbol{\epsilon}}}^{\top}(l;\nu)\hat{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon}}^{-1}(\nu)\mathbf{C}_{\hat{\boldsymbol{\epsilon}}}(l;\nu)\hat{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon}}^{-1}(\nu-l)\}, \tag{4.28}$$

$$\mathcal{Q}_M^{*}(\nu) = N\sum_{l=1}^{M}\frac{N}{N-\lfloor(l-\nu+s)/s\rfloor}\mathrm{tr}\{\mathbf{C}_{\hat{\boldsymbol{\epsilon}}}^{\top}(l;\nu)\hat{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon}}^{-1}(\nu)\mathbf{C}_{\hat{\boldsymbol{\epsilon}}}(l;\nu)\hat{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon}}^{-1}(\nu-l)\} \tag{4.29}$$

Under the null hypothesis of adequacy of a particular PVAR(p) model, the test statistics $\mathcal{Q}_M(\nu)$ and $\mathcal{Q}_M^{*}(\nu)$ follow approximatively a chi-square distribution

$\chi^2_{d^2(M-p)}$, where χ^2_d denotes a chi-square distribution with d degrees of freedom. As discussed in McLeod (1994) in the univariate case and confirmed in the simulation experiments of Ursu and Duchesne (2009) in the multivariate framework, the Ljung-Box correction factor $N/\{N - \lfloor (l - \nu + s)/s \rfloor\}$ generally improves the finite sample properties of the test statistic (4.28), where $\lfloor x \rfloor$ represents the integer part of the real number x. Global test statistics can be used to test the null hypothesis of model adequacy for all seasons taken simultaneously. These test statistics are constructed by summing (4.28) or (4.29) over all seasons :

$$\mathcal{Q}_M = \sum_{\nu=1}^{s} \mathcal{Q}_M(\nu), \tag{4.30}$$

$$\mathcal{Q}^*_M = \sum_{\nu=1}^{s} \mathcal{Q}^*_M(\nu). \tag{4.31}$$

In the class of PVAR time series models of order p for each season, the distributions of the test statistics $\mathcal{Q}_M$ and $\mathcal{Q}^*_M$ are approximatively chi-square with $d^2 s(M - p)$ degrees of freedom. These properties follow since the test statistics (4.28) and (4.29) are asymptotically independent across the seasons $\nu = 1, \ldots, s$. These global test statistics represent natural extensions of proposals originally suggested by Hipel and McLeod (1994, p. 500) in univariate PAR models.

Here, we consider the test statistics (4.28), (4.29), (4.30) and (4.31) for checking SPVAR time series models, which under the null hypothesis are supposed to be generated by (4.1), with specified orders p_1 and p_2. To justify their use in that class of models, we first establish the asymptotic distributions of $\mathbf{c}_{\hat{\boldsymbol{\epsilon}}}(\nu)$ and $\mathbf{r}_{\hat{\boldsymbol{\epsilon}}}(\nu)$, $\nu = 1, \ldots, M$, based on the SPVAR residuals $\hat{\boldsymbol{\epsilon}}_t$, $t = 1, \ldots, n$. The developments invoke arguments similar to those needed to derive the asymptotic distributions of the residual autocovariance and autocorrelation matrices in PVAR time series models, see Ursu and Duchesne (2009). However, complications arise due to the inherent non-linearity in the parameters of SPVAR models. Furthermore, to study the asymptotic distributions of the portmanteau test statistics, the asymptotic covariance matrices of the residual autocovariance matrices are needed ; as showed below, the asymptotic matrices for SPVAR models may be very different that those established in the PVAR class.

First, remark that the vector of sample autocovariances $N^{1/2}\mathbf{c}_{\boldsymbol{\epsilon}}(\nu)$ follows asymptotically a normal distribution of dimension d^2M :

$$N^{1/2}\mathbf{c}_{\boldsymbol{\epsilon}}(\nu) \xrightarrow{d} N_{d^2M}(\mathbf{0}, \mathbf{V}(\nu; M) \otimes \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu)), \tag{4.32}$$

where $\mathbf{V}(\nu; M)$ corresponds to the $dM \times dM$ block diagonal matrix :

$$\mathbf{V}(\nu; M) = \begin{pmatrix} \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu-1) & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu-2) & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & & & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu-M) \end{pmatrix}. \tag{4.33}$$

See Ursu and Duchesne (2009). From (4.26), we deduce that :

$$\lim_{N\to\infty} N\text{cov}\{\hat{\boldsymbol{\beta}}(\nu) - \boldsymbol{\beta}(\nu), \mathbf{c}_{\boldsymbol{\epsilon}}(\nu)\} = -\boldsymbol{\Omega}^{-1}(\nu) \lim_{N\to\infty} E\left[\mathbf{S}_N\{\boldsymbol{\beta}(\nu)\}\mathbf{c}_{\boldsymbol{\epsilon}}^{\top}(\nu)\right].$$

Given the definition of $\mathbf{S}_N\{\boldsymbol{\beta}(\nu)\}$, and using the fact that

$$\mathbf{c}_{\boldsymbol{\epsilon}}^{\top}(l;\nu) = N^{-1}\sum_{n=l}^{N-1} \boldsymbol{\epsilon}_{ns+\nu-l}^{\top} \otimes \boldsymbol{\epsilon}_{ns+\nu}^{\top},$$

we obtain :

$$\lim_{N\to\infty} E\left[\mathbf{S}_N\{\boldsymbol{\beta}(\nu)\}\mathbf{c}_{\boldsymbol{\epsilon}}^{\top}(l;\nu)\right] =$$

$$\lim_{N\to\infty} N^{-1}\sum_{n=0}^{N-1}\sum_{t=l}^{N-1} E\left[\{\boldsymbol{\Delta}_{ns+\nu}^{\top}\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\nu)\boldsymbol{\epsilon}_{ns+\nu}\}\{\boldsymbol{\epsilon}_{ts+\nu-l}^{\top} \otimes \boldsymbol{\epsilon}_{ts+\nu}^{\top}\}\right] = \mathbf{H}(l;\nu).$$

Using the infinite moving average representation (4.7), an explicit expression for the $(p_1+p_2)d^2 \times d^2$ matrix $\mathbf{H}(l;\nu)$ is given by the following expression :

$$\mathbf{H}^{\top}(l;\nu) = \left(\mathbf{H}_{11}^{\top}(l;\nu), \dots, \mathbf{H}_{p_21}^{\top}(l;\nu), \mathbf{H}_{12}^{\top}(l;\nu), \dots, \mathbf{H}_{p_12}^{\top}(l;\nu)\right),$$

where $\mathbf{H}_{11}(l;\nu), \dots, \mathbf{H}_{p_21}(l;\nu)$ and $\mathbf{H}_{12}(l;\nu), \dots, \mathbf{H}_{p_12}(l;\nu)$ denote matrices of dimension $d^2 \times d^2$. These matrices are defined by :

$$\begin{aligned} \mathbf{H}_{i1}(l;\nu) &= \boldsymbol{\Psi}_{l-i}(\nu-i)\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu-l) \otimes \mathbf{I}_d - \boldsymbol{\Psi}_{l-s-i}(\nu-i)\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu-l) \otimes \boldsymbol{\Lambda}_1^{\top}(\nu) \\ &\quad - \dots - \boldsymbol{\Psi}_{l-p_1s-i}(\nu-i)\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu-l) \otimes \boldsymbol{\Lambda}_{p_1}^{\top}(\nu),\ i = 1, \dots, p_2, \\ \mathbf{H}_{i2}(l;\nu) &= \boldsymbol{\Psi}_{l-is}(\nu)\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu-l) \otimes \mathbf{I}_d - \boldsymbol{\Phi}_1(\nu)\boldsymbol{\Psi}_{l-is-1}(\nu-1)\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu-l) \otimes \mathbf{I}_d \\ &\quad - \dots - \boldsymbol{\Phi}_{p_2}(\nu)\boldsymbol{\Psi}_{l-is-p_2}(\nu-p_2)\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}(\nu-l) \otimes \mathbf{I}_d,\ i = 1, \dots, p_1, \end{aligned}$$

where $\mathbf{\Psi}_0(\nu) = \mathbf{I}_d$ and $\mathbf{\Psi}_k(\nu) = \mathbf{0}$ for $k < 0$, $\nu = 1, \ldots, s$. Collecting $\mathbf{H}(l;\nu)$, $l = 1, \ldots, M$, in a $(p_1 + p_2)d^2 \times Md^2$ matrix leads to the expression :

$$\mathbf{H}(\nu) = (\mathbf{H}(1;\nu), \ldots, \mathbf{H}(M;\nu)) . \tag{4.34}$$

Consequently, we deduce that the asymptotic distributions of the residual autocovariance matrices are given by :

$$N^{1/2}\mathbf{c}_{\hat{\boldsymbol{\epsilon}}}(\nu) \xrightarrow{d} N_{d^2M}(\mathbf{0}, \mathbf{\Xi}(\nu)) , \ \nu = 1, \ldots, s, \tag{4.35}$$

where $\mathbf{\Xi}(\nu) = \mathbf{V}(\nu; M) \otimes \mathbf{\Sigma}_{\boldsymbol{\epsilon}}(\nu) - \mathbf{H}^\top(\nu)\mathbf{\Omega}^{-1}(\nu)\mathbf{H}(\nu)$. Due to the non-linearity in the parameters of SPVAR time series models, the asymptotic distribution of the residual autocovariance matrices is different than the one found in PVAR models ; this is seen in the form of the asymptotic covariance matrix of (4.35), through the matrix $\mathbf{H}(\nu)$ defined by (4.34).

Let

$$\mathbf{L}(\nu) = \begin{pmatrix} \mathbf{\Gamma}_0^{-1}(\nu-1) & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{\Gamma}_0^{-1}(\nu-2) & \mathbf{0} & \ldots & \mathbf{0} \\ \vdots & & & \ddots & \vdots \\ \mathbf{0} & & & \ldots & \mathbf{\Gamma}_0^{-1}(\nu-M) \end{pmatrix} \otimes \mathbf{\Gamma}_0^{-1}(\nu).$$

Since $\mathbf{D}_{\hat{\boldsymbol{\epsilon}}}(\nu)$ converges to $\mathbf{\Gamma}_0(\nu)$ in probability, that is $\mathbf{D}_{\hat{\boldsymbol{\epsilon}}}(\nu) \xrightarrow{p} \mathbf{\Gamma}_0(\nu)$, the asymptotic distribution of $n^{1/2}\mathbf{r}_{\hat{\boldsymbol{\epsilon}}}(\nu)$ is obtained by an appropriate scaling of result (4.35). More precisely, the asymptotic distributions of the residual autocorrelation matrices are given by :

$$N^{1/2}\mathbf{r}_{\hat{\boldsymbol{\epsilon}}}(\nu) \xrightarrow{d} N_{d^2M}(\mathbf{0}, \mathbf{L}(\nu)\mathbf{\Xi}(\nu)\mathbf{L}^\top(\nu)), \ \nu = 1, \ldots, s.$$

In order to study the distribution of the portmanteau test statistics, we introduce the $d^2 \times d^2$ matrices $\mathbf{P}_i$, $i = 1, \ldots, M$, satisfying :

$$\begin{aligned} \mathbf{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\nu - i) \otimes \mathbf{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\nu) &= \mathbf{P}_i^\top \mathbf{P}_i, \ i = 1, \ldots, M, \\ \mathbf{P}_i\{\mathbf{\Sigma}_{\boldsymbol{\epsilon}}(\nu - i) \otimes \mathbf{\Sigma}_{\boldsymbol{\epsilon}}(\nu)\}\mathbf{P}_i^\top &= \mathbf{I}_{d^2}. \end{aligned}$$

Let the block-diagonal matrix $\mathbf{Q}_M$ be defined by :

$$\mathbf{Q}_M = \begin{pmatrix} \mathbf{P}_1 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_2 & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & & & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{P}_M \end{pmatrix}, \tag{4.36}$$

and consider $\tilde{\mathbf{c}}_{\hat{\boldsymbol{\epsilon}}}(\nu) = \mathbf{Q}_M \mathbf{c}_{\hat{\boldsymbol{\epsilon}}}(\nu)$. It follows that the asymptotic covariance matrix of $\tilde{\mathbf{c}}_{\hat{\boldsymbol{\epsilon}}}(\nu)$ is given by :

$$\lim_{N\to\infty} N\text{var}\{\tilde{\mathbf{c}}_{\hat{\boldsymbol{\epsilon}}}(\nu)\} = \mathbf{I}_{d^2M} - \mathbf{Q}_M \left\{\mathbf{H}^\top(\nu)\boldsymbol{\Omega}^{-1}(\nu)\mathbf{H}(\nu)\right\} \mathbf{Q}_M^\top.$$

Using the relation $\mathbf{Q}_M^\top \mathbf{Q}_M = \mathbf{V}^{-1}(\nu; M) \otimes \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\nu)$, and reworking the arguments established in PVAR time series models, the following relation holds approximately :

$$\boldsymbol{\Omega}(\nu) \approx \mathbf{H}(\nu)\{\mathbf{V}^{-1}(\nu; M) \otimes \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}^{-1}(\nu)\}\mathbf{H}^\top(\nu), \tag{4.37}$$

for $M > p_1 s + p_2$. From the previous relation, we deduce that the asymptotic covariance matrix of $\tilde{\mathbf{c}}_{\hat{\boldsymbol{\epsilon}}}(\nu)$ is approximatively idempotent.

The asymptotic distribution (4.35) is useful to test the joint significance of $\mathbf{c}_{\hat{\boldsymbol{\epsilon}}}(l;\nu)$, $l = 1, \dots, M$, for a given M and a given season $\nu \in \{1, \dots, s\}$. The test statistic $\mathcal{Q}_M(\nu)$ satisfies the relations :

$$\begin{aligned} \mathcal{Q}_M(\nu) &= N \sum_{l=1}^{M} \text{tr}\{\mathbf{C}_{\hat{\boldsymbol{\epsilon}}}^\top(l;\nu)\hat{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon}}^{-1}(\nu)\mathbf{C}_{\hat{\boldsymbol{\epsilon}}}(l;\nu)\hat{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon}}^{-1}(\nu-l)\}, \\ &= N\mathbf{c}_{\hat{\boldsymbol{\epsilon}}}^\top(\nu)\left\{\hat{\mathbf{V}}^{-1}(\nu; M) \otimes \hat{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon}}^{-1}(\nu)\right\}\mathbf{c}_{\hat{\boldsymbol{\epsilon}}}(\nu), \\ &= N\mathbf{c}_{\hat{\boldsymbol{\epsilon}}}^\top(\nu)\hat{\mathbf{Q}}_M^\top\hat{\mathbf{Q}}_M\mathbf{c}_{\hat{\boldsymbol{\epsilon}}}(\nu). \end{aligned}$$

In the last equality, by construction, $\hat{\mathbf{Q}}_M$ provides a consistent estimator of $\mathbf{Q}_M$. Thus, the portmanteau test statistics $\mathcal{Q}_M(\nu)$ and $\mathcal{Q}_M^*(\nu)$ have approximatively a chi-square distribution $\chi^2_{d^2(M-p_1-p_2)}$ under the null hypothesis of adequacy, in the class of SPVAR time series models. Furthermore, the global test statistics $\mathcal{Q}_M$ and $\mathcal{Q}_M^*$ follow approximatively a chi-square distribution with $d^2 s(M - p_1 - p_2)$ degrees of freedom.

In a Technical Report (available in Appendix), the test statistics proposed in this section were illustrated in a small empirical study. The following test statistics

were included in our experiments : the portmanteau test statistics calculated at each season, that is $\mathcal{Q}_M(\nu)$ and $\mathcal{Q}_M^*(\nu)$, $\nu = 1, \ldots, s$, and also the global versions $\mathcal{Q}_M$ and $\mathcal{Q}_M^*$. To compare the exact distributions of the test statistics with their corresponding χ^2 distributions, two bivariate SPVAR(1,1) stochastic process with a periodic Gaussian white noise were used. We considered the case of quarterly data by setting $\nu = 4$. We examined the empirical frequencies of rejection of the null hypothesis of adequacy at two different nominal levels (5 and 10 percent) for each of three series lengths ($N = 400$, $N = 600$ and $N = 800$ observations by season). For each series length, 10000 independent realizations were generated. For each realization of the data generating process, a SPVAR(1,1) model was estimated using the least squares estimation technique, as described in Section 4.3. For each residual time series, the portmanteau test statistics $\mathcal{Q}_M(\nu)$, $\mathcal{Q}_M^*(\nu)$, the global portmanteau test statistics $\mathcal{Q}_M$ and $\mathcal{Q}_M^*$ were calculated for $M = 10, 15, 20, 25, 30, 35, 40, 45, 50$. Finally, for each nominal level and for each series of length $n = 4N$, we obtained the empirical frequencies of rejection of the null hypothesis of adequacy. We now summarize the results.

(a) As expected, the test statistics $\mathcal{Q}_M^*(\nu)$ exhibited better empirical levels than the uncorrected version $\mathcal{Q}_M(\nu)$. As for PVAR time series models, the correction factor proposed by McLeod (1994) improved the χ^2 approximation for the test statistic $\mathcal{Q}_M^*(\nu)$, offering generally better finite sample properties than $\mathcal{Q}_M(\nu)$, particularly for large values of M.

(b) In view of point (1), we concentrate the rest of our discussion on $\mathcal{Q}_M^*(\nu)$ only. At the 5% and 10% nominal levels, slight over-rejections have been observed for the small lag order $M = 10$. To use the correction factor did not improve the finite sample behavior, and the exact distributions of the portmanteau test statistics $\mathcal{Q}_M^*(\nu)$ appeared to be relatively far from the asymptotic distributions. To increase the sample size to $N = 800$ did not offer significantly better finite sample performance. In view of the theory elaborated in this section, this phenomenon seems to occur because (4.37) holds approximately, and the approximation appears to be more satisfactory for large M (in fact, one has to approximate an infinite sum by a

finite sum in order to establish (4.37)). Since $p_1 = p_2 = 1$ in our experiments, the approximation may be poor if M appears to be close of $p_1 s + p_2 = 1 \times 4 + 1 = 5$. That observation has been confirmed in other simulation experiments. Generally, the χ^2 distribution provided a reasonable approximation for large M and taking $M > 10$ offered satisfactory empirical levels at both significance levels, at least in our empirical study.

(c) As the test statistics calculated at each season, the global portmanteau test statistic $\mathcal{Q}_M^*$ displayed better empirical levels than the uncorrected test statistic $\mathcal{Q}_M$. Furthermore, some over-rejections has been observed for very small lags. Generally, the empirical performance of (4.31) appeared to be more satisfactory than the uncorrected version (4.30). In general, the global test statistics $\mathcal{Q}_M^*$ with $M > 15$ offered reasonable empirical rejections, particularly for large values of N.

From our limited empirical study, the finite sample performance of the portmanteau test statistics seemed rather reasonable, particularly for moderate to large sample sizes. Given the number of parameters involved in vector periodic time series, it is not really surprising that moderate to large sample sizes are needed in order to have satisfactory results. Overall, the test statistics $\mathcal{Q}_M^*(\nu)$, $\nu = 1, \ldots, s$ and $\mathcal{Q}_M^*$ with moderate to large values of M can be recommended for diagnosing SPVAR time series models.

4.5. Application using a bivariate quarterly data set on international travellers entering or returning to Canada

In this section, the proposed methodology is illustrated with two univariate time series data coming from the Canadian Socio-Economic Information Management System of Statistics Canada. The bivariate data set is composed of the total number of international travellers entering or returning to two provinces of Canada : New Brunswick and Manitoba. New Brunswick is the nearest Eastern province to both Central Canada and the United States, while Manitoba, due to

its position in Canada, acts as a northern hub of the mid-continent trade corridor. Several trade missions have been undertaken by these two provinces in order to develop their markets at the international scale. These data provide economic indicators of a certain type of tourism activities in these provinces. The period ranges from January 1972 to September 2007. Given the nature of the data, the sample size, and for the purpose of our illustration, the original monthly data were converted to quarterly data by aggregation (thus the period $\nu = 4$ has been naturally selected). The original data are represented in Figure 4.1.

Generally, the variability of the univariate time series seemed to decrease as a function of the time unit. Thus, in order to stabilize the variance, the Box-Cox transformation has been considered. For each variable, zero was included in the 95% confidence interval for the transformation parameter, or zero was very close to that interval, suggesting a logarithmic transformation. The time series data are displayed season by season in Figure 4.2. Generally, the variable corresponding to the international travellers entering to New Brunswick displayed

FIGURE 4.1. Total number of international travellers entering or returning to New Brunswick (top) and Manitoba (bottom), from January 1972 to September 2007. The original monthly data were converted to quarterly data by aggregation.

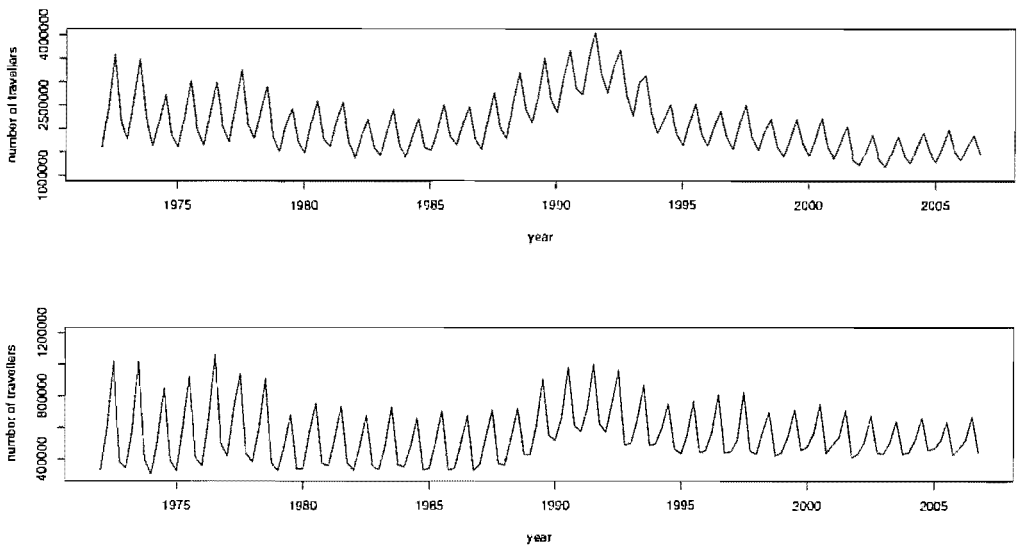
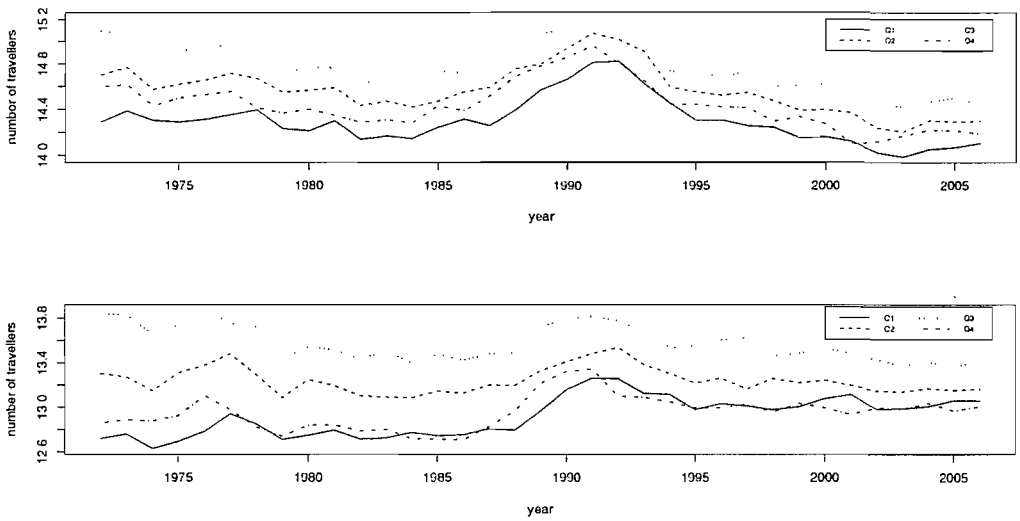


FIGURE 4.2. Total number of international travellers entering or returning to New Brunswick (top) and Manitoba (bottom), by season. The quarters of a given year are naturally numbered 1 to 4. The data were transformed using the logarithmic transformation.



a downward trend, which was more or less pronounced (except for the period 1986-1996 : international tourism experienced a period of rapid growth worldwide, until the 1997 Asian financial crisis, which affected the international tourism industry. That growth occurred in Canada and was particularly perceptible in New Brunswick. See the Statistics Canada document entitled *Travel-log*, Vol. 18, Number 1, 1999. In Figure 1, the downward trend dominating the whole series does not seem strongly affected). For the international travellers visiting Manitoba, we found a slight upward trend and a similar growth in the period 1986-1996, but of smaller magnitude. In our application, the trends were more perceptible on the logarithmic scale.

Consequently, it may be relevant to test for the presence of trends, and to check if they are stochastic or deterministic. In view of this, we implemented the methodology of Franses and Paap (2004, Section 4), which is summarized in their Table 4.3. For each variable, a general univariate PAR model has been estimated, imposing the same order for each season. The order $p = 5$ has been postulated

in each season, which has been gradually reduced using the portmanteau test statistics for univariate PAR models developed in McLeod (1994). Using the 5% nominal level, a PAR(4) has been estimated for the international travellers visiting New Brunswick, while the largest PAR(5) model has been retained for the international travellers visiting Manitoba.

All the roots of the characteristic equations of the PAR(p) models written in the so-called 'vector of quarters form' have been calculated, see Franses and Paap (2004, p. 35). The smallest roots (in module) for the international travellers visiting New Brunswick and Manitoba were 1.19 and 1.61, respectively. For each variable, the smallest root appeared to be single. Since these estimated roots were reasonably close to one, the presence of unit roots has been expected. Likelihood ratio tests were undertaken in order to find evidence of unit roots. Consider the following PAR(p) model for quarterly data :

$$Y_{4n+\nu} = \mu(\nu) + \tau(\nu)T_{4n+\nu} + \sum_{k=1}^{p} \phi_k(\nu) Y_{4n+\nu-k} + \epsilon_{4n+\nu},$$

where $\mu(\nu)$, $\nu \in \{1,2,3,4\}$, denote trending parameters; $T_t = \lfloor (t-1)/4 \rfloor$ represents the linear deterministic trend (which is constant across the year n) with coefficients $\tau(\nu)$, $\nu \in \{1,2,3,4\}$; $\phi_k(\nu)$, $\nu \in \{1,2,3,4\}$, denote the autoregressive coefficients which are defined periodically, that is $\phi_k(\nu - 4k) = \phi_k(\nu)$, $k \in \mathbb{N}$, $\nu \in \{1,2,3,4\}$; and $\epsilon_{4n+\nu}$ corresponds to the error term. A quarterly time series is said to be periodically integrated of order one when $1 - \alpha(\nu)B$ is needed in order to remove the stochastic trend from $\{Y_{4n+\nu}\}$, where the coefficients $\alpha(\nu)$, $\nu \in \{1,2,3,4\}$, are seasonally-varying parameters with the property that $\prod_{\nu=1}^{4} \alpha(\nu) = 1$. The filter $1 - \alpha(\nu)B$ is called a periodic differencing filter associated to season ν, $\nu \in \{1,2,3,4\}$. See Osborn (1988) or Franses and Paap (2004). The null hypothesis of periodic integration, that is $\prod_{\nu=1}^{4} \alpha(\nu) = 1$, has been tested for each univariate time series. We used the test statistics for periodic integration introduced in Boswijk and Franses (1996, p. 228). See also Franses and Paap (2004, pp. 78-79). For the international travellers visiting New Brunswick and Manitoba, the values of the test statistics were 0.027 and 3.113, respectively. Using the critical values given in Table 10.A.2 of Fuller (1996, p. 642), the null

hypotheses could not be rejected and the presence of unit roots is concluded for each time series.

The inclusion of linear deterministic trends in periodically-integrated autoregressive (PIAR) time series models assumes the presence of seasonal quadratic trends in the original time series. In order to test which deterministic trends have to be included in the time series, it is often more informative to test jointly for the null hypothesis of periodic integration, that is $\prod_{\nu=1}^{4} \alpha(\nu) = 1$, and for the absence of a quadratic trend, see Paap and Franses (1999, p. 281) and Franses and Paap (2004, p. 80). For the international travellers visiting New Brunswick and Manitoba, the values of the test statistics were 4.420 and 3.714, respectively. Using the critical values in Table B.4 of Franses and Paap (2004, p. 128), the null hypotheses could not be rejected, suggesting in each time series a unit root and the absence of a quadratic trend.

In PIAR time series models, it appears relevant to test for the presence of no quadratic trends (NQT), common seasonal linear trends (CLT) and no linear trends (NLT), using the testing strategy of Franses and Paap (2004, Section 4.2). The test statistics LR_{NQT} and $LR_{\tau=0}$ represent useful tools to investigate the nature of deterministic trends. On the other hand, $LR_{\mu=\tau=0}$ and LR_{NLT} allow us to test for linear deterministic trends, and the test statistic LR_{CLT} checks the possibility of common linear deterministic trends. These test statistics are precisely defined in Paap and Franses (1999). The results are presented in Table 4.1. At the 1% nominal level, the absence of quadratic trends is confirmed (however, at the 5% nominal level, the hypothesis is rejected for the total number of international travellers entering to New Brunswick). The null hypothesis $\tau(\nu) = 0, \nu \in \{1, 2, 3, 4\}$ is rejected, as indicated by the test statistic $LR_{\tau=0}$. Furthermore, the test statistics reject the absence of deterministic trends : the P-values of the test statistics $LR_{\mu=\tau=0}$ are smaller than any reasonable significance level and those of LR_{NLT} are smaller than the 5% nominal level. The common linear trend hypothesis is not rejected at the 5% nominal level, as suggested by the results of the likelihood test statistic LR_{CLT}.

TABLE 4.1. Testing for periodic integration and restrictions on the deterministic components. The P-values corresponding to the test statistics are given in parentheses. The notation 0^+ denotes a number inferior to 10^{-3}.

	LR_{NQT}	$LR_{\tau=0}$	LR_{CLT}	LR_{NLT}	$LR_{\mu=\tau=0}$
Number of international travellers	4.393	25.844	9.297	27.647	30.904
entering to New Brunswick;	(0.036)	(0^+)	(0.054)	(0^+)	(0^+)
Number of international travellers	0.601	16.718	8.286	13.628	23.311
entering to Manitoba;	(0.438)	(0.002)	(0.082)	(0.018)	(0.003)

1. LR_{NQT} denotes the likelihood ratio test statistic for the absence of quadratic deterministic trends;
2. LR_{CLT} is the likelihood ratio test statistic for checking common deterministic trend across the seasons;
3. LR_{NLT} represents the likelihood ratio test statistic for the absence of linear deterministic trends;
4. $LR_{\tau=0}$ and $LR_{\mu=\tau=0}$ are likelihood ratio test statistics for checking $\tau(\nu) = 0$, $\nu \in \{1,2,3,4\}$ and $\mu(\tau) = \tau(\nu) = 0$, $\nu \in \{1,2,3,4\}$, respectively.

Once it has been established that the null hypothesis of periodic integration cannot be rejected, it may be interesting to investigate whether the following hypotheses

$$\begin{aligned} H_0^{(a)} : \alpha(\nu) &= 1, \quad \nu \in \{1,2,3\}, \\ H_0^{(b)} : \alpha(\nu) &= -1, \quad \nu \in \{1,2,3\}, \end{aligned}$$

are valid (note that under $\prod_{\nu=1}^{4} \alpha(\nu) = 1$ this implies that $\alpha(4) = 1$ or $\alpha(4) = -1$, respectively). Under $H_0^{(a)}$, the periodic differencing filter reduces to the first difference $1 - B$, and the periodic autoregressive model contains a non-seasonal unit root. However, under $H_0^{(b)}$, the filter becomes $1 + B$, corresponding to the seasonal unit root -1. See also Boswijk and Franses (1996, p. 231) or Franses and Paap (2004, p. 81). For the international travellers visiting New Brunswick and Manitoba, the values of the test statistics (with the P-values in parentheses) were 12.517 (0.0058) and 9.4752 (0.0236), respectively. Consequently, at the 1%

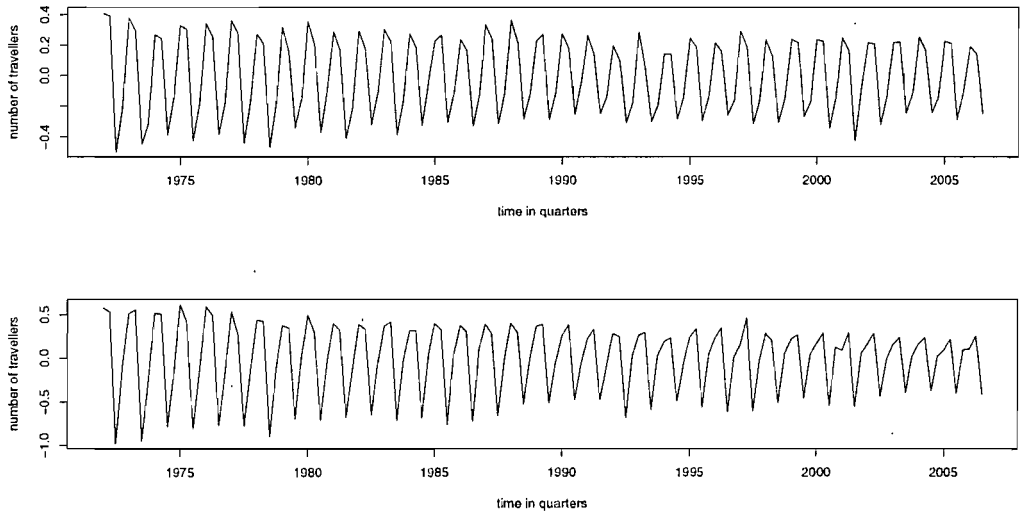
nominal level, the non-periodic differencing filter $1-B$ seemed appropriate for the international travellers visiting Manitoba, but the null hypothesis is rejected for New Brunswick. At the 5% nominal level, the null hypotheses are rejected for each univariate time series. For the international travellers visiting New Brunswick, we found $(\hat{\alpha}(1), \hat{\alpha}(2), \hat{\alpha}(3), \hat{\alpha}(4)) = (1.009, 0.952, 0.879, 1.184)$ and for those visiting Manitoba we obtained $(\hat{\alpha}(1), \hat{\alpha}(2), \hat{\alpha}(3), \hat{\alpha}(4)) = (0.798, 0.878, 1.036, 1.377)$. Note that the null hypotheses $H_0^{(b)}$ were clearly rejected at any reasonable significance level for each time series, a conclusion which was naturally expected, given the estimators of $\alpha(\nu)$, $\nu \in \{1, 2, 3, 4\}$.

A cointegration analysis has been performed, following the method presented in Franses and Paap (2004, p. 112). After regressing the variable corresponding to the international travellers entering to New Brunswick on those entering Manitoba, the residuals have been represented graphically season by season ; since they did not display a stationary behavior (downward trends were clearly perceptible for most of the residual time series), we concluded that no serious evidence of cointegration was present. We performed a more formal analysis using the test statistics of Engle and Granger (1987) ; the results have confirmed the graphical analysis. Diagnostic checking cointegrated vector periodic time series models represents a topic of current research, which is outside the scope of the present paper ; consequently, we do not pursue in that direction in the present paper.

The time series data obtained by applying the first difference to each variable are represented in Figure 4.3, and by season in Figure 4.4. Quite obviously, strong seasonal patterns are observed, year after year. The graphics presented in Figure 4.4 are advocated in Paap and Franses (2004, p. 14) ; they represent useful tools to investigate the nature of the seasonality. In our application, the lines corresponding to the seasons are rather distinct for several seasons, and generally this appears to be true for each data set. For the international travellers visiting New Brunswick, the lines for the third and fourth quarters are separated, with no intersection with the others, while the lines of the first and second quarters display many intersections between them. For the international travellers visiting Manitoba, a similar behavior is observed : the lines associated to quarters one

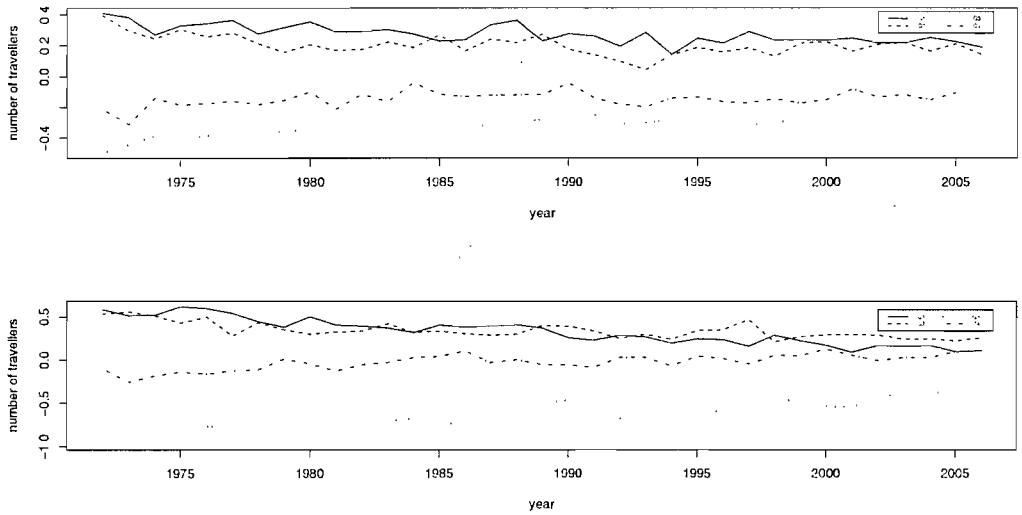
and two offer many intersections, with essentially no intersection with the line of quarter four, which is close to those lines, and the line corresponding to the third quarter is far from the others.

FIGURE 4.3. Total number of international travellers entering or returning to New Brunswick (top) and Manitoba (bottom), from January 1972 to September 2007. The data have been transformed by applying the first difference of the logarithm for each variable.



Overall, in order to have stationarity, the time series data on the international travellers visiting New Brunswick and Manitoba have been transformed using the logarithm function. Concerning the choice of the differencing filter, three situations have been investigated : first, we applied the periodic differencing filter $1 - \alpha(\nu)B$ on each variable, since at the 5% nominal level the null hypothesis that $\alpha(\nu) = 1, \nu \in \{1, 2, 3, 4\}$ was rejected for both variables. However, at the 1% nominal level the null hypothesis was not rejected for the international travellers visiting Manitoba; consequently, we investigated the usual filter $1 - B$ for these travellers. From a forecasting point of view, the usual $1 - B$ filter may delivers in some occasions better forecasting performance than the periodic differencing filter, see Franses and Paap (2004, pp. 95-97). In view of this, the time series data

FIGURE 4.4. Total number of international travellers entering or returning to New Brunswick (top) and Manitoba (bottom), by season. The quarters of a given year are naturally numbered 1 to 4. The data have been transformed by applying the first difference of the logarithm for each variable.



were analyzed using the non-periodic differencing filter $1 - B$ for both variables. With these filters, the sample size was equal to $n = 136$.

A (non-periodic) VAR model for the variables differentiated using the $1 - B$ filter, and with autoregressive order $p = 5$, has been adjusted to the bivariate time series. According to Lütkepohl (2005, p. 602), this order appears to be a reasonable choice, because each observation is explained by the quarters of the last year, plus an additional quarter of the previous year. A residual analysis has been made and the portmanteau test statistics of Hosking (1980) and Li and McLeod (1981) have been calculated, without and with the Ljung-Box adjustment. They are defined as :

$$\mathcal{P}_M = n \sum_{l=1}^{M} \mathrm{tr}\{\mathbf{C}_{\hat{\boldsymbol{\epsilon}}}^{\top}(l)\hat{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon}}^{-1}\mathbf{C}_{\hat{\boldsymbol{\epsilon}}}(l)\hat{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon}}^{-1}\}, \tag{4.38}$$

$$\mathcal{P}_M^{\star} = n \sum_{l=1}^{M} \frac{n}{n-l} \mathrm{tr}\{\mathbf{C}_{\hat{\boldsymbol{\epsilon}}}^{\top}(l)\hat{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon}}^{-1}\mathbf{C}_{\hat{\boldsymbol{\epsilon}}}(l)\hat{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon}}^{-1}\}, \tag{4.39}$$

where $\mathbf{C}_{\hat{\boldsymbol{\epsilon}}}^{\top}(l)$, $|l| < n$ are the residual autocovariance matrices and $\hat{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon}}$ corresponds to a consistent estimator of the covariance matrix of the error term under the null hypothesis that a VAR model is appropriate. The test statistics (4.38) and (4.39) are described in Lütkepohl (2005); they follow approximatively a $\chi^2_{d^2(M-p)}$ distribution under the null hypothesis of model adequacy. Here, $d = 2$, $p = 5$. The adjustment of a VAR(5) model gave P-values smaller than any reasonable significance level for our bivariate time series data, suggesting clearly an inappropriate model.

Since the bivariate data set display strong seasonal patterns, PVAR models (with possibly different orders in each season) have been adjusted. Before to estimate the periodic time series models, the seasonal means and trends were removed from the time series, using the methods and models discussed in Franses and Paap (2004, Chapter 4). We tried to adjust PVAR models of small orders to the time series data. When the periodic differencing filters were applied to both variables, PVAR models with small seasonal orders were inappropriate at the 5% nominal level, in at least one season. When the periodic differencing filter was applied only to the international travellers visiting New Brunswick, and using the usual first difference for those visiting Manitoba, PVAR models with small and moderate seasonal orders were rejected at the 5% nominal level, in at least one season. In the third situation, where the first difference filter was applied to both variables, a similar phenomenon occurred : PVAR models with small and moderate orders were inappropriate. However, it should be noted that when the large orders $(p(1), p(2), p(3), p(4)) = (3, 3, 3, 3)$ were selected, PVAR(3) models seemed satisfactory in the three situations, since all the P-values of the portmanteau test statistics did not indicate inappropriate models, at the 5% nominal level. However, these models were not particularly parsimonious (the number of estimated parameters were 48, which is rather large, given the available sample size).

TABLE 4.2. P-values of the portmanteau test statistics defined by (4.28) and (4.29) in adjusting the time series data on the travellers entering or returning to New Brunswick and Manitoba, using a bivariate SPVAR(1,1) model with $\nu = 4$. The periodic differencing filter is applied to each time series.

	$Q_M(\nu)$				$Q_M^*(\nu)$			
M \ ν	1	2	3	4	1	2	3	4
12	0.7711	0.2942	0.3495	0.2048	0.6802	0.2049	0.2688	0.1649
16	0.8211	0.6902	0.3631	0.2787	0.6873	0.5568	0.2293	0.1950
20	0.9060	0.7218	0.4086	0.2217	0.7731	0.5213	0.2218	0.1053
24	0.9449	0.6021	0.4621	0.1107	0.8046	0.3034	0.2128	0.0259

TABLE 4.3. P-values of the portmanteau test statistics defined by (4.28) and (4.29) in adjusting the time series data on the travellers entering or returning to New Brunswick and Manitoba, using a bivariate SPVAR(1,1) model with $\nu = 4$. The periodic differencing filter is applied to the international travellers entering or returning to New Brunswick and a first difference filter is applied to Manitoba.

	$Q_M(\nu)$				$Q_M^*(\nu)$			
M \ ν	1	2	3	4	1	2	3	4
12	0.2315	0.8171	0.2937	0.6082	0.1593	0.7477	0.2194	0.5501
16	0.5028	0.9549	0.3178	0.6533	0.3591	0.9137	0.1942	0.5472
20	0.8199	0.9134	0.5331	0.2338	0.6766	0.7922	0.3392	0.0996
24	0.9059	0.9632	0.5625	0.4291	0.7578	0.8588	0.3024	0.1990

In view of the previous results, we tried to find parsimonious representations using SPVAR time series models. Consequently, for each time series data, SPVAR(1,1) models have been adjusted. For quarterly data, these time series models rely on 32 independent parameters. The P-values of the portmanteau test

TABLE 4.4. P-values of the portmanteau test statistics defined by (4.28) and (4.29) in adjusting the time series data on the travellers entering or returning to New Brunswick and Manitoba, using a bivariate SPVAR(1,1) model with $\nu = 4$. The data have been transformed by applying the first difference of the logarithm for each variable.

	$Q_M(\nu)$				$Q_M^*(\nu)$			
M \ ν	1	2	3	4	1	2	3	4
12	0.7653	0.5417	0.5528	0.0419	0.6720	0.4475	0.4685	0.0283
16	0.9735	0.7641	0.6799	0.1231	0.9443	0.6447	0.5484	0.0768
20	0.8644	0.7784	0.4096	0.2976	0.6845	0.5959	0.2073	0.1852
24	0.9467	0.8188	0.4497	0.5514	0.8072	0.5806	0.1934	0.3703

statistics for diagnosing the estimated models are presented in Tables 4.2, 4.3 and 4.4. From the simulation experiments reported in Section 4.4, we observed some over-rejections for low values of M. In view of that empirical evidence, we report the P-values of the portmanteau test statistics for moderate to large values of M, more precisely for $M = 12, 16, 20$ and 24.

From Table 4.2, all the P-values suggest that the SPVAR(1,1) model is not rejected at the usual 5% significance level, except for $Q_{24}^*(4)$, with an empirical significance level of about 2.6%. However, this does not strongly point to model inadequacy since that empirical level is still not too low, and, consequently, the possibility of an error of type I exists. Based on the observed P-values, PVAR models with small seasonal orders were also rejected at the 5% nominal level. However, in that particular situation, a PVAR(2) model relying also on 32 parameters could be a close competitor to the adjusted SPVAR(1,1) model, since both models offered generally a satisfying adjustment. Since from a forecasting point of view the first difference filter may be more efficient than the periodic differencing filter (see Franses and Paap (2004, pp. 95-97)), we report in Table 4.3 the results when the periodic differencing filter is applied only to the international

travellers visiting New Brunswick, and $1 - B$ is considered for the international travellers visiting Manitoba. In Table 4.4, the P-values of the adjustment using the first difference filter for both variables are given. From the P-values reported in Table 4.3, the SPVAR model seems appropriate, since all the P-values are superior to the 5% nominal level. From Table 4.4, where the first difference filter has been applied to both variables, all the P-values suggest that the SPVAR(1,1) model is not rejected at the usual 5% significance level, except for the case $M = 12$, where the P-value of $Q^*_{12}(4)$ is about 2.8%, which does not strongly suggest model inadequacy. In these two particular situations PVAR(2) time models were clearly inappropriate at the 5% nominal level. Interestingly, these results suggest that the usual $1 - B$ filter may be useful, at least for these data.

Overall, these adjustments indicate that SPVAR models seem reasonable for these travellers time series data. Our analyses suggest that SPVAR time series models may be useful in practice, providing under certain circumstances more parsimonious representations than PVAR models (if found adequate, they may be particularly useful for time series whose sample sizes are not very large). Given the complexity of multivariate models, and the number of parameters involved in vector periodic models, the class of SPVAR models should offers a useful complement to PVAR modeling.

4.6. Conclusion

In this paper, we introduced a new class of multivariate seasonal models with periodically varying parameters, called SPVAR models. In that class of models, the asymptotic distributions of the least squares estimators have been obtained. Furthermore, we derived the asymptotic distributions of the residual autocovariance matrices in the class of SPVAR models, and the asymptotic distributions of the residual autocorrelation matrices have been established as a corollary. As applications of our asymptotic results, portmanteau test statistics have been studied, including test statistics for each season and also global versions. The advocated test statistics were evaluated in a small simulation study, whose results where briefly discussed (the complete results are available in a Technical Report). From

our simulation experiments, the test statistics with the Ljung-Box adjustment performed reasonably well and can be recommended for use. The methodology has been applied successfully to a bivariate time series, composed of quarterly data, on travellers entering or returning to certain provinces of Canada. Overall, it is hoped that the results presented in this paper will be useful in practical applications, complementing PVAR modeling, especially in finding and diagnosing more parsimonious representations of periodic vector time series models.

4.7. Acknowledgements

This work is part of the Ph.D. thesis of the first author under the supervision of the second author. The authors would like to thank the Editor P. H. Franses, an Associate Editor and a referee for their constructive remarks and suggestions. Many thoughtful comments of the referees led to significant improvements of the paper. This research is supported by a grant from the National Science and Engineering Research Council of Canada.

4.8. Appendix : Simulation Experiments

In the paper we studied test statistics for diagnosing SPVAR models. From a practical point of view, it is natural to inquire about the finite-sample properties of these test statistics, in particular their exact levels. To partially answer that question, we report in this appendix the simulation results of a small Monte Carlo experiment. The following test statistics are included in our experiments : the portmanteau test statistics calculated at each season, that is $\mathcal{Q}_M(\nu)$ and $\mathcal{Q}^*_M(\nu)$, $\nu = 1, \ldots, s$, and also the global versions $\mathcal{Q}_M$ and $\mathcal{Q}^*_M$. To compare the exact distributions of the test statistics with their corresponding χ^2 distributions, the following bivariate data generating process (DGP) was used :

$$\begin{aligned} \text{DGP}_i : \mathbf{Y}_{ns+\nu} &= \mathbf{\Phi}_i(\nu)\mathbf{Y}_{ns+\nu-1} + \mathbf{\Lambda}_i(\nu)\mathbf{Y}_{(n-1)s+\nu} - \\ &\quad \mathbf{\Lambda}_i(\nu)\mathbf{\Phi}_i(\nu)\mathbf{Y}_{(n-1)s+\nu-1} + \boldsymbol{\epsilon}_{ns+\nu},\ i = 1, 2, \end{aligned} \tag{4.40}$$

where the process $\boldsymbol{\epsilon} = \{\boldsymbol{\epsilon}_t, t \in \mathbb{Z}\}$ was assumed a periodic Gaussian white noise, composed of independent Gaussian random vectors with mean $\mathbf{0}$ and covariance

matrix $\mathbf{\Sigma}_{\boldsymbol{\epsilon},i}(\nu)$, $i = 1, 2$. The DGP generated by (4.40) corresponds to a SPVAR(1,1) stochastic process. We considered the case of quarterly data by setting $\nu = 4$. The model parameter matrices $\mathbf{\Phi}_i(\nu)$, $\mathbf{\Lambda}_i(\nu)$, $\mathbf{\Sigma}_{\boldsymbol{\epsilon},i}(\nu)$, $i = 1, 2$, $\nu = 1, \ldots, 4$ in DGP_1 and DGP_2 are given in Tables 4.5 and 4.6, respectively. It may be verified that both models satisfy the causality condition. Under DGP_1, six roots of the equation (4.6) are real numbers and four complex numbers, while under DGP_2, all the roots of the equation (4.6) are real numbers.

We examined the empirical frequencies of rejection of the null hypothesis of adequacy at two different nominal levels (5 and 10 percent) for each of three series lengths ($N = 400$, $N = 600$ and $N = 800$ observations by season). For each series length, 10000 independent realizations were generated. For each realization of the DGP_i, $i = 1, 2$, a SPVAR(1,1) model was estimated using the least squares estimation technique, as described in Section 4.3.

For each residual time series, the portmanteau test statistics $\mathcal{Q}_M(\nu)$ and $\mathcal{Q}^*_M(\nu)$ and the global portmanteau test statistics $\mathcal{Q}_M$ and $\mathcal{Q}^*_M$ were calculated for $M = 10, 15, 20, 25, 30, 35, 40, 45, 50$. For each nominal level and for each series of length $n = 4N$, we obtained from the 10000 realizations the empirical

TABLE 4.5. Model parameter matrices for DGP_1.

$$\mathbf{\Phi}_1(1) = \begin{pmatrix} 0.19 & 0.10 \\ 0.22 & 0.10 \end{pmatrix}, \quad \mathbf{\Phi}_1(2) = \begin{pmatrix} 0.31 & 0.13 \\ -0.30 & 0.40 \end{pmatrix},$$

$$\mathbf{\Phi}_1(3) = \begin{pmatrix} -0.70 & 0.10 \\ 0.49 & 0.60 \end{pmatrix}, \quad \mathbf{\Phi}_1(4) = \begin{pmatrix} -0.41 & 0.39 \\ 0.80 & -0.30 \end{pmatrix},$$

$$\mathbf{\Lambda}_1(1) = \begin{pmatrix} 0.30 & 0.10 \\ 0.20 & 0.30 \end{pmatrix}, \quad \mathbf{\Lambda}_1(2) = \begin{pmatrix} -0.20 & 0.30 \\ -0.10 & -0.20 \end{pmatrix},$$

$$\mathbf{\Lambda}_1(3) = \begin{pmatrix} 0.50 & 0.60 \\ 0.10 & 0.53 \end{pmatrix}, \quad \mathbf{\Lambda}_1(4) = \begin{pmatrix} 0.45 & 0.10 \\ 0.10 & 0.20 \end{pmatrix},$$

$$\mathbf{\Sigma}_{\boldsymbol{\epsilon},1}(1) = \begin{pmatrix} 5.00 & 2.00 \\ 2.00 & 7.00 \end{pmatrix}, \quad \mathbf{\Sigma}_{\boldsymbol{\epsilon},1}(2) = \begin{pmatrix} 12.00 & 4.00 \\ 4.00 & 2.00 \end{pmatrix},$$

$$\mathbf{\Sigma}_{\boldsymbol{\epsilon},1}(3) = \begin{pmatrix} 8.00 & 1.00 \\ 1.00 & 6.00 \end{pmatrix}, \quad \mathbf{\Sigma}_{\boldsymbol{\epsilon},1}(4) = \begin{pmatrix} 7.00 & 5.00 \\ 5.00 & 9.00 \end{pmatrix}.$$

TABLE 4.6. Model parameter matrices for DGP_2.

$$\mathbf{\Phi}_2(1) = \begin{pmatrix} -0.95 & 0.15 \\ 0.54 & 0.62 \end{pmatrix}, \quad \mathbf{\Phi}_2(2) = \begin{pmatrix} 0.72 & 0.21 \\ -0.30 & 0.45 \end{pmatrix},$$

$$\mathbf{\Phi}_2(3) = \begin{pmatrix} 0.24 & 0.16 \\ 0.27 & 0.15 \end{pmatrix}, \quad \mathbf{\Phi}_2(4) = \begin{pmatrix} -0.36 & 0.44 \\ 0.85 & -0.25 \end{pmatrix},$$

$$\mathbf{\Lambda}_2(1) = \begin{pmatrix} 0.61 & 0.21 \\ 0.44 & 0.12 \end{pmatrix}, \quad \mathbf{\Lambda}_2(2) = \begin{pmatrix} 0.43 & 0.12 \\ 0.31 & 0.37 \end{pmatrix},$$

$$\mathbf{\Lambda}_2(3) = \begin{pmatrix} 0.35 & -0.31 \\ -0.47 & -0.52 \end{pmatrix}, \quad \mathbf{\Lambda}_2(4) = \begin{pmatrix} -0.20 & 0.51 \\ 0.27 & 0.55 \end{pmatrix},$$

$$\mathbf{\Sigma}_{\boldsymbol{\epsilon},2}(1) = \begin{pmatrix} 1.60 & 0.30 \\ 0.30 & 1.00 \end{pmatrix}, \quad \mathbf{\Sigma}_{\boldsymbol{\epsilon},2}(2) = \begin{pmatrix} 0.20 & 0.10 \\ 0.10 & 0.80 \end{pmatrix},$$

$$\mathbf{\Sigma}_{\boldsymbol{\epsilon},2}(3) = \begin{pmatrix} 1.00 & 0.30 \\ 0.30 & 0.50 \end{pmatrix}, \quad \mathbf{\Sigma}_{\boldsymbol{\epsilon},2}(4) = \begin{pmatrix} 0.50 & 0.10 \\ 0.10 & 0.20 \end{pmatrix}.$$

frequencies of rejection of the null hypothesis of adequacy. Based on 10000 realizations, acceptable empirical levels at the 5% and 10% nominal levels should be in the intervals $[4.57, 5.43]$ and $[9.41, 10.59]$, respectively.

The empirical levels of the portmanteau test statistics $\mathcal{Q}_M(\nu)$ and $\mathcal{Q}_M^*(\nu)$ for DGP_1 and DGP_2 are presented in Tables 4.7 and 4.8, respectively. As expected, the test statistics $\mathcal{Q}_M^*(\nu)$ exhibited better empirical levels than the uncorrected version $\mathcal{Q}_M(\nu)$. As for PVAR time series models, the correction factor proposed by McLeod (1994) improved the χ^2 approximation for the test statistic $\mathcal{Q}_M^*(\nu)$, offering generally better finite sample properties than $\mathcal{Q}_M(\nu)$, particularly for large values of M. In view of this, we concentrate the rest of our discussion on $\mathcal{Q}_M^*(\nu)$ only.

At the 5% and 10% nominal levels, some over-rejections have been observed for the small lag order $M = 10$: for DGP_1 at $\nu = 3$, and for DGP_2 for all seasons $\nu = 1, \ldots, 4$. To use the correction factor did not improve the finite sample behavior, and the exact distributions of the portmanteau test statistics appeared to be relatively far from the asymptotic distributions. To increase the sample size to $N = 800$ did not offer significantly better finite sample performance. In

TABLE 4.7. Empirical levels (in percentage) of the portmanteau test statistics $Q_M(\nu)$ defined by (4.28), and its modified version $Q_M^*(\nu)$ defined by (4.29), for the SPVAR model generated by (4.40) and the coefficients given in Table 4.5.

	$N = 400$															
	$\alpha = 0.05$								$\alpha = 0.10$							
	$Q_M(\nu)$				$Q_M^*(\nu)$				$Q_M(\nu)$				$Q_M^*(\nu)$			
M \ ν	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
10	5.03	4.89	5.56	4.97	5.28	5.03	5.76	5.17	10.36	9.94	11.85	9.85	10.85	10.38	12.19	10.04
15	4.18	4.83	5.11	4.43	4.52	5.19	5.37	4.71	9.41	9.73	10.12	9.32	10.17	10.43	10.67	9.76
20	3.87	4.67	4.51	4.46	4.51	5.22	5.05	4.86	8.63	9.48	9.19	9.31	9.66	10.42	10.06	10.10
25	3.95	4.52	4.60	3.98	4.63	5.32	5.21	4.64	8.18	9.30	9.12	8.62	9.43	10.67	10.29	10.08
30	3.71	4.39	4.16	4.13	4.69	5.30	5.10	4.92	7.97	8.78	8.61	8.44	9.64	10.30	10.06	9.62
35	3.35	4.27	4.08	3.99	4.27	5.27	5.10	4.99	7.14	8.63	8.08	8.42	9.08	10.41	9.79	9.86
40	3.35	4.13	3.67	3.90	4.58	5.43	4.83	5.16	6.98	8.49	8.05	7.92	9.20	10.44	9.98	10.16
45	2.93	3.72	3.65	3.65	4.31	5.26	4.80	4.91	6.48	7.84	7.32	7.83	8.96	10.45	9.98	10.00
50	2.81	3.52	3.46	3.59	4.42	5.19	5.16	5.18	6.35	7.60	7.23	7.53	9.20	10.48	9.99	10.12
	$N = 600$															
	$\alpha = 0.05$								$\alpha = 0.10$							
	$Q_M(\nu)$				$Q_M^*(\nu)$				$Q_M(\nu)$				$Q_M^*(\nu)$			
M \ ν	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
10	5.24	4.97	5.39	5.25	5.44	5.09	5.57	5.36	10.25	10.20	11.57	10.69	10.47	10.43	11.85	10.86
15	4.53	5.13	5.16	5.14	4.88	5.32	5.46	5.43	9.59	9.92	10.52	10.10	10.00	10.31	10.88	10.49
20	4.21	4.73	4.88	5.08	4.57	5.04	5.28	5.29	9.08	9.59	10.04	9.89	9.84	10.26	10.75	10.38
25	3.90	4.55	4.80	5.03	4.35	5.08	5.05	5.55	8.35	9.39	9.44	9.96	9.35	10.36	10.37	10.77
30	3.87	4.42	4.44	4.50	4.43	5.08	5.07	5.00	8.09	8.87	9.38	9.91	9.09	9.78	10.30	10.86
35	3.77	4.07	4.35	4.55	4.56	4.85	5.06	5.14	7.97	8.64	9.02	9.42	9.45	9.88	10.51	10.50
40	3.59	4.11	3.77	4.03	4.41	5.11	4.69	4.88	7.85	8.58	8.53	8.76	9.26	10.02	10.14	10.19
45	3.43	3.88	3.88	4.13	4.37	4.84	4.90	5.10	7.38	8.33	8.46	8.67	9.28	10.04	10.07	10.26
50	3.25	3.78	3.86	4.21	4.41	4.97	4.95	5.20	7.40	8.11	8.18	8.28	9.31	10.12	10.21	10.23
	$N = 800$															
	$\alpha = 0.05$								$\alpha = 0.10$							
	$Q_M(\nu)$				$Q_M^*(\nu)$				$Q_M(\nu)$				$Q_M^*(\nu)$			
M \ ν	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
10	4.99	5.06	5.83	5.29	5.10	5.22	5.93	5.40	10.28	10.27	11.67	11.07	10.52	10.40	11.84	11.17
15	4.77	4.57	5.14	4.93	4.94	4.80	5.31	5.13	9.49	10.03	10.44	10.28	9.85	10.36	10.77	10.53
20	4.70	4.97	4.97	4.81	5.12	5.17	5.14	5.16	9.57	9.82	9.72	9.95	10.04	10.25	10.18	10.18
25	4.34	4.47	4.55	4.66	4.69	4.87	4.86	4.98	9.16	9.26	9.55	9.62	9.70	9.97	10.10	10.27
30	4.31	4.48	4.43	4.43	4.73	4.90	4.88	4.83	8.73	9.22	9.33	9.46	9.61	9.99	10.07	10.20
35	4.37	4.31	4.54	4.32	4.86	4.86	5.07	4.87	8.45	8.90	9.26	8.98	9.48	10.04	10.16	9.91
40	4.25	4.43	4.29	4.60	4.99	5.08	4.93	5.02	8.33	8.81	9.13	8.84	9.46	10.12	10.25	9.96
45	3.61	4.07	4.42	4.16	4.35	4.82	5.09	4.92	8.22	8.53	8.88	8.80	9.85	10.06	10.15	9.92
50	3.62	4.04	4.35	4.23	4.53	4.98	5.08	5.04	7.88	8.48	8.49	8.72	9.56	10.20	9.73	10.05

TABLE 4.8. Empirical levels (in percentage) of the portmanteau test statistics $Q_M(\nu)$ defined by (4.28), and its modified version $Q_M^*(\nu)$ defined by (4.29), for the SPVAR model generated by (4.40) and the coefficients given in Table 4.6.

	$N = 400$															
	$\alpha = 0.05$								$\alpha = 0.10$							
	$Q_M(\nu)$				$Q_M^*(\nu)$				$Q_M(\nu)$				$Q_M^*(\nu)$			
M \ ν	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
10	5.99	5.17	5.99	5.79	6.32	5.40	6.24	5.96	11.54	10.59	11.80	11.75	12.14	11.00	12.26	12.05
15	4.42	4.94	4.92	4.93	4.90	5.34	5.24	5.18	9.82	10.23	10.07	10.31	10.57	10.99	10.67	10.78
20	4.19	4.76	4.36	4.73	4.64	5.23	4.82	5.27	8.72	9.60	9.34	9.74	9.75	10.68	10.16	10.57
25	4.09	4.49	4.63	4.53	4.76	5.13	5.25	5.08	8.34	9.25	9.17	8.89	9.59	10.50	10.16	10.08
30	3.80	4.42	4.33	4.27	4.54	5.53	5.17	5.10	8.13	9.02	8.68	8.85	9.62	10.27	10.30	10.16
35	3.66	4.13	4.12	4.17	4.68	5.41	5.01	5.30	7.39	8.69	8.14	8.62	9.27	10.42	9.88	10.42
40	3.41	3.86	3.80	3.86	4.72	5.25	5.03	5.17	7.17	8.32	7.77	7.99	9.40	10.69	9.89	10.23
45	2.98	3.71	3.54	3.57	4.31	5.28	4.77	4.87	6.72	7.59	7.46	7.76	9.23	10.25	9.98	9.97
50	2.84	3.49	3.34	3.47	4.31	5.21	4.88	5.08	6.37	7.49	7.20	7.41	9.15	10.52	9.87	10.46
	$N = 600$															
	$\alpha = 0.05$								$\alpha = 0.10$							
	$Q_M(\nu)$				$Q_M^*(\nu)$				$Q_M(\nu)$				$Q_M^*(\nu)$			
M \ ν	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
10	6.01	5.85	6.31	6.44	6.23	6.04	6.52	6.61	11.86	11.30	12.52	12.30	12.11	11.43	12.70	12.43
15	5.05	5.21	5.18	5.71	5.44	5.47	5.40	5.93	10.20	10.47	10.60	10.86	10.84	10.83	11.10	11.43
20	4.48	4.65	4.88	5.34	4.87	5.05	5.22	5.63	9.13	9.98	9.81	10.25	9.83	10.63	10.35	10.81
25	4.23	4.69	4.55	4.96	4.72	4.99	5.15	5.50	8.50	9.57	9.62	10.03	9.36	10.40	10.18	10.68
30	3.67	4.47	4.50	4.92	4.37	5.11	5.16	5.41	8.46	9.11	9.22	9.67	9.60	9.93	10.15	10.77
35	3.74	4.18	3.97	4.56	4.62	4.97	4.62	5.31	8.17	8.86	8.68	9.51	9.40	10.08	9.99	10.61
40	3.69	4.16	3.94	4.47	4.58	5.00	4.68	5.20	8.04	8.83	8.58	8.71	9.58	10.45	10.08	10.24
45	3.57	4.11	3.70	4.31	4.59	5.10	4.75	5.19	7.71	8.44	8.34	8.76	9.41	10.41	10.06	10.48
50	3.49	3.88	4.01	4.01	4.73	5.03	5.02	5.17	7.46	8.19	8.04	8.27	9.29	10.24	10.02	10.23
	$N = 800$															
	$\alpha = 0.05$								$\alpha = 0.10$							
	$Q_M(\nu)$				$Q_M^*(\nu)$				$Q_M(\nu)$				$Q_M^*(\nu)$			
M \ ν	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
10	5.68	5.48	6.11	5.88	5.80	5.62	6.19	5.95	11.46	10.59	11.91	11.77	11.74	10.77	12.11	11.87
15	4.95	4.87	5.55	5.11	5.15	5.10	5.75	5.28	10.22	10.08	10.77	10.46	10.59	10.44	10.99	10.74
20	4.71	4.62	5.10	4.74	4.97	4.89	5.37	4.96	9.39	9.63	10.39	9.89	9.85	9.94	10.81	10.31
25	4.41	4.67	4.75	4.96	4.80	4.90	5.18	5.21	8.73	9.18	9.95	9.77	9.48	9.79	10.53	10.32
30	4.47	4.21	4.86	4.59	4.84	4.82	5.31	4.91	8.86	9.36	9.70	9.20	9.70	10.24	10.49	9.95
35	4.12	4.29	4.86	4.47	4.79	4.85	5.39	4.94	8.83	9.03	9.58	9.02	9.84	9.89	10.54	9.80
40	4.15	4.49	4.55	4.33	4.81	4.93	5.17	5.10	8.21	8.47	9.23	8.97	9.40	9.63	10.31	10.06
45	3.89	4.31	4.46	4.07	4.71	5.00	5.28	4.77	8.18	8.63	9.17	8.52	9.56	9.80	10.46	9.75
50	3.71	4.39	4.39	4.11	4.65	5.22	5.37	4.79	8.05	8.67	8.86	8.27	9.54	10.19	10.38	9.50

view of the theory elaborated in Section 4.4, this phenomenon seems to occur because (4.37) holds approximately, and the approximation appears to be more satisfactory for large M (in fact, one has to approximate an infinite sum by a finite sum in order to establish (4.37)). Since $p_1 = p_2 = 1$, the approximation may be poor if M appears to be close of $s + 1 = 5$. That observation has been confirmed in unreported simulation experiments. Generally, the χ^2 distribution provided a reasonable approximation for large M and taking $M \geq 15$ offered satisfactory empirical levels at both significance levels, at least in our empirical study.

The empirical levels of the global portmanteau tests for SPVAR models are presented in Tables 4.9 and 4.10. As the test statistics calculated at each season, the test statistic Q_M^* displayed better empirical levels than the uncorrected test statistic Q_M. Furthermore, some over-rejections has been observed for lags $M = 10$ under DGP_1, and for $M = 10, 15$ under DGP_2. Generally, the empirical performance of (4.31) appeared to be more satisfactory than the uncorrected

TABLE 4.9. Empirical levels (in percentage) of the global portmanteau test Q_M and Q_M^* defined by (4.30) and (4.31) for the SPVAR model generated by (4.40) and the coefficients given in Table 4.5.

	$N = 400$				$N = 600$				$N = 800$			
	$\alpha = 0.05$		$\alpha = 0.10$		$\alpha = 0.05$		$\alpha = 0.10$		$\alpha = 0.05$		$\alpha = 0.10$	
M	Q_M	Q_M^*	Q_M	Q_M^*	Q_M	Q_M^*	Q_M	Q_M^*	Q_M	Q_M^*	Q_M	Q_M^*
10	5.84	6.31	11.61	12.27	5.65	6.04	11.47	11.95	5.96	6.22	11.93	12.22
15	4.72	5.46	9.78	11.30	4.81	5.25	9.90	10.56	5.07	5.44	10.31	10.91
20	4.38	5.32	8.89	10.65	4.50	5.18	9.45	10.54	4.60	5.18	9.58	10.48
25	3.92	5.26	8.41	10.87	4.30	5.17	8.71	10.41	4.11	4.94	8.94	10.32
30	3.62	5.27	7.76	10.55	3.92	4.96	8.03	10.02	4.07	4.85	8.38	10.04
35	3.30	5.55	7.40	10.44	3.39	4.72	7.47	9.89	3.99	5.16	8.24	9.93
40	2.97	5.47	6.79	10.86	3.27	4.76	7.09	9.98	3.84	4.95	8.02	10.02
45	2.80	5.35	5.88	10.52	3.21	4.78	6.64	10.12	3.57	4.84	7.74	10.33
50	2.38	5.17	5.22	10.38	2.98	4.85	6.30	9.94	3.37	5.21	7.62	10.33

TABLE 4.10. Empirical levels (in percentage) of the global portmanteau test Q_M and Q_M^* defined by (4.30) and (4.31) for the SPVAR model generated by (4.40) and the coefficients given in Table 4.6.

	$N = 200$				$N = 400$				$N = 600$			
	$\alpha = 0.05$		$\alpha = 0.10$		$\alpha = 0.05$		$\alpha = 0.10$		$\alpha = 0.05$		$\alpha = 0.10$	
M	Q_M	Q_M^*	Q_M	Q_M^*	Q_M	Q_M^*	Q_M	Q_M^*	Q_M	Q_M^*	Q_M	Q_M^*
10	7.06	7.58	13.88	14.71	7.44	7.82	14.06	14.60	6.95	7.15	13.04	13.49
15	5.25	6.09	10.71	12.12	5.61	6.20	11.23	12.17	5.30	5.68	10.72	11.34
20	4.46	5.53	9.04	10.83	4.75	5.36	9.43	10.74	4.92	5.37	9.84	10.64
25	4.07	5.44	8.45	10.88	4.44	5.23	9.09	10.70	4.47	5.32	9.21	10.53
30	3.78	5.28	7.67	11.05	4.08	5.23	8.56	10.52	4.29	5.20	8.86	10.22
35	3.59	5.56	7.31	10.84	3.77	5.26	7.85	10.31	4.21	5.18	8.38	10.18
40	2.98	5.31	6.67	10.72	3.44	5.21	7.42	10.22	4.07	5.40	8.15	10.15
45	2.70	5.16	5.89	10.75	3.09	4.97	6.82	10.31	3.89	5.22	7.70	9.86
50	2.35	5.14	5.24	10.49	2.87	4.84	6.42	10.09	3.46	4.95	7.31	10.10

version (4.30). In general, the global test statistics Q_M^* with $M \geq 20$ offered empirical rejections in the intervals $[4.57, 5.43]$ and $[9.41, 10.59]$ at the 5% and 10% nominal levels, respectively, or very close to these intervals, particularly for large values of N.

From this limited empirical study, the finite sample performance of the portmanteau test statistics seems rather reasonable, particularly for moderate to large sample sizes. Given the number of parameters involved in vector periodic time series, it is not really surprising that moderate to large sample sizes are needed in order to have satisfactory results. Overall, the test statistics $Q_M^*(\nu)$, $\nu = 1, \ldots, s$ and Q_M^* with moderate to large values of M can be recommended for diagnosing SPVAR time series models.

4.9. REFERENCES

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CONCLUSION

Nous avons présenté quelques résultats concernant les séries chronologiques multivariées saisonnières et périodiques. Nous allons résumer les résultats les plus importants et indiquer quelques avenues possibles de recherche. Les principaux objectifs de cette thèse ont porté sur l'ajustement de modèles vectoriels parcimonieux à des données saisonnières. Plus exactement, on s'est intéressé à l'estimation des paramètres et à la vérification de la qualité de l'ajustement pour des modèles saisonniers ou périodiques.

Dans la première partie de la thèse, on s'est intéressé aux modèles saisonniers vectoriels (SVARMA). Contrairement aux modèles saisonniers univariés qui ont été largement étudiés, les modèles saisonniers vectoriels ont reçu moins d'attention. À notre connaissance, nous sommes les premiers à avoir développé des résultats asymptotiques pour les estimateurs des paramètres et pour les matrices d'autocovariance résiduelles pour ce type de modèle. Nous avons considéré aussi la situation où il existe des contraintes linéaires sur les paramètres. Bien que les résultats soient semblables à ceux obtenus pour les modèles VARMA avec des contraintes ou les modèles SARIMA avec des contraintes, certaines difficultés surviennent compte tenu de la non-linéarité dans les paramètres de modèles SVARMA. Ainsi, nos résultats généralisent la littérature existante dans les deux orientations précisées ci-dessus. On s'attend que les modèles saisonniers vectoriels, qui sont des compétiteurs pour les modèles périodiques vectoriels, soient plus utiles quand le nombre d'observations n'est pas très grand. En conséquence, le développement proposé pour les modèles saisonniers vectoriels semble justifié.

Dans la deuxième partie de la thèse, les principales propriétés d'un processus PVAR sont présentées. On rappelle qu'un modèle PVAR peut être exprimé

comme un modèle VAR qui est appelé processus empilé. Grâce à cette propriété, il est important de souligner que les conditions de stationnarité (au sens périodique) d'un processus PVAR peuvent être obtenues en utilisant le processus empilé correspondant. L'utilisation d'un modèle PVAR capable de représenter une série observée comporte un certain nombre d'étapes. Il s'agit notamment de l'identification du modèle (de l'ordre), de l'estimation des paramètres et de la validation avec des tests de type portemanteau. La méthode des moindres carrés est utilisée afin d'obtenir les estimateurs des paramètres du modèle PVAR et leur comportement asymptotique. Comme les processus multivariés périodiques impliquent un nombre important de paramètres indépendants, nous considérons les situations où il existe des contraintes linéaires sur les paramètres d'une saison. Nous trouvons la distribution asymptotique des matrices d'autocovariances et d'autocorrélations résiduelles dans le cadre des modèles PVAR. Une étude de simulation montre le comportement de la distribution asymptotique des matrices d'autocovariances résiduelles. Nous avons utilisé des données de l'Allemagne de l'Ouest afin d'illustrer la procédure d'estimation des paramètres et le test portemanteau sur les résidus. Cet exemple illustre que le modèle PVAR avec des contraintes linéaires sur les paramètres est utile pour la modélisation de séries chronologiques périodiques.

Finalement, nous introduisons un modèle multivarié saisonnier autorégressif avec des paramètres périodiques (SPVAR). Le modèle est utile pour des données multivariées, et combine une structure périodique autorégressive et un modèle multiplicatif saisonnier. Les mêmes étapes que dans le cas du modèle PVAR sont respectées. Cependant, l'estimation et le test diagnostic sont plus compliqués que dans le cas des modèles PVAR étant donné la non-linéarité dans les paramètres. La méthodologie a été appliquée à une série chronologique bivariée, composée de données trimestrielles, sur les voyageurs qui entrent ou qui reviennent dans certaines provinces du Canada. Globalement, il est espéré que les résultats présentés dans le chapitre 4 seront utiles dans les applications, complétant les résultats dans les modèles PVAR. Particulièrement, on s'attend que les modèles PVAR d'ordres élevés pourront être remplacés par des modèles SPVAR plus parcimonieux.

Comme première avenue de recherche, nous pourrions nous intéresser au problème de l'identification de modèles vectoriels périodiques autorégressifs (PVAR) (ou modèles périodiques en général) avec un nombre minimal de paramètres. L'estimation des paramètres d'un modèle autorégressif périodique exige que, simultanément, les ordres autorégressifs et les paramètres soient déterminés en fonction de certains critères d'optimisation. Comme règle générale, différents critères de sélection des modèles tels que le critère AIC et le critère BIC peuvent être utilisés afin de faciliter le choix le plus approprié du modèle. Pour l'objectif d'identification, chaque saison peut être modélisée indépendamment de toute autre saison. Nous pourrions utiliser un critère séparé pour chaque saison (ce qui est utilisé actuellement dans la littérature). Mais cela n'implique pas automatiquement la minimisation d'un critère global, car généralement le minimum d'une somme de fonctions est différent de la somme des minimums. Nous pouvons utiliser les algorithmes génétiques pour calculer les sous-modèles VAR (voir Baragona, Battaglia et Cucina (2004)). L'identification des sous-modèles VAR a été largement étudiée dans la littérature. Des algorithmes efficaces qui ne considèrent pas tous les sous-modèles possibles peuvent être aussi utilisés comme dans Gatu et Kontoghiorghes (2005).

Une autre avenue de recherche d'intérêt porte sur le test portemanteau pour les modèles VAR périodiques cointégrés en utilisant les propriétés des matrices d'autocovariances résiduelles (voir par exemple Brüggemann, Lütkepohl et Saikkonen (2006)). Plusieurs approches peuvent être utilisées. Une des approches consiste à utiliser la représentation VAR d'un modèle PVAR ce qui pourrait être intéressant d'un point de vue théorique, mais est problématique en raison du grand nombre de paramètres en cause. Une solution pourrait être l'usage de matrices bloc-diagonales impliquées dans cette représentation (voir Ghysels et Osborn (2001)). Une autre approche a été proposée par Boswijk et Franses (1995) reposant sur le modèle à correction d'erreurs. Franses et Kloek (1995) ont utilisé la méthodologie en deux étapes de Engle et Granger (1987). Cette approche est une approche saison par saison qui ne peut pas étudier l'ensemble des relations de cointégration.

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